Algebraic aspects of symbolic dynamics 2026 - July 13-17, 2026 - Porto

Odomutants and flexibility results for quantitative orbit equivalence

Corentin Correia

Université Paris Cité

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Examples:

- **Irrational rotation** of angle θ:

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- Finite alphabet Σ, prob. measure ν on Σ.
  The **Bernoulli shift** on (Σ,ν):

$$T\colon (x_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}},$$

$$\mu = \nu^{\otimes\mathbb{Z}}.$$

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$$\begin{array}{rccl} S\colon & X & \longrightarrow & X \\ & (x_n)_{n \geq 0} & \longmapsto & (x_n)_{n \geq 0} + (1, 0, 0, 0, \ldots) \text{ with carry over to right} \end{array}$$

$S(q_0 - 1, q_1 - 1, q_2 - 1, \ldots) = (0, 0, 0, \ldots).$

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But in full generality, the conjugacy problem is very hard!

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Conjugacy asks for $\Psi \in \mathrm{Aut}(X,\mu)$ such that $\Psi(Tx) = S\Psi(x)$

$$\begin{tikzcd}
\cdots & \bullet \arrow[r, "T"] \arrow[d, "\Psi"] & \bullet \arrow[r, "T"] \arrow[d, "\Psi"] & \bullet \arrow[r, "T"] \arrow[d, "\Psi"] & \bullet \arrow[r, "T"] \arrow[d, "\Psi"] & \bullet \arrow[r, "T"] \arrow[d, "\Psi"] & \bullet \arrow[d, "\Psi"] & \cdots \\
\cdots & \bullet \arrow[r, "S"'] & \bullet \arrow[r, "S"'] & \bullet \arrow[r, "S"'] & \bullet \arrow[r, "S"'] & \bullet \arrow[r, "S"'] & \bullet & \cdots
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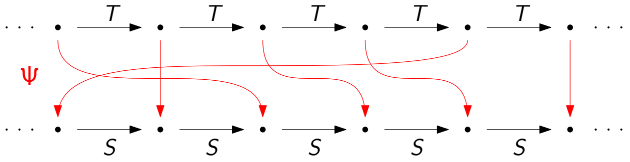
Definition (Dye 1959)

T and $S \in \mathrm{Aut}(X,\mu)$ are **orbit equivalent (OE)** if there exists $\Psi \in \mathrm{Aut}(X,\mu)$ such that $\Psi(\mathrm{Orb}_T(x)) = \mathrm{Orb}_S(\Psi(x))$ forr μ-a.e. $x \in X$.



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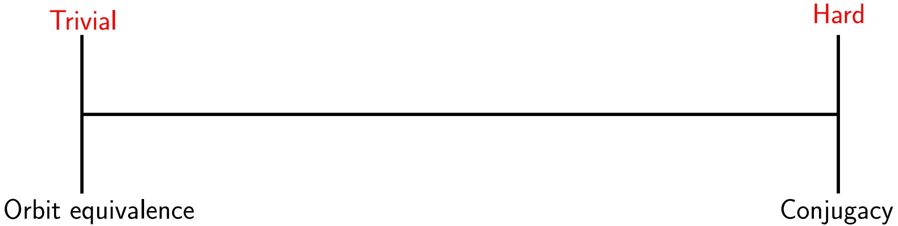

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$c_T \colon X \to \mathbb{Z}$ and $c_S \colon X \to \mathbb{Z}$ are the **cocycles** associated to the orbit equivalence Ψ.

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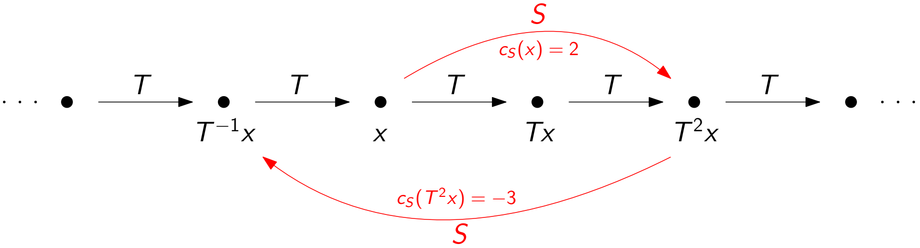
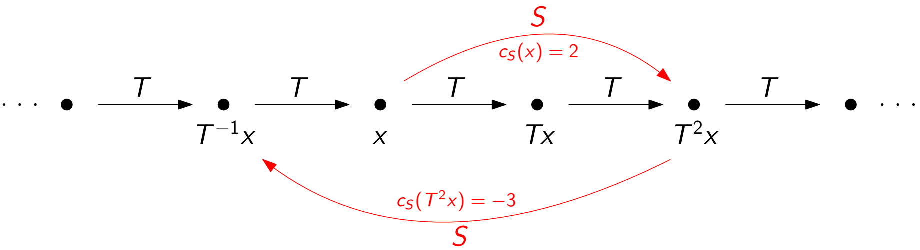
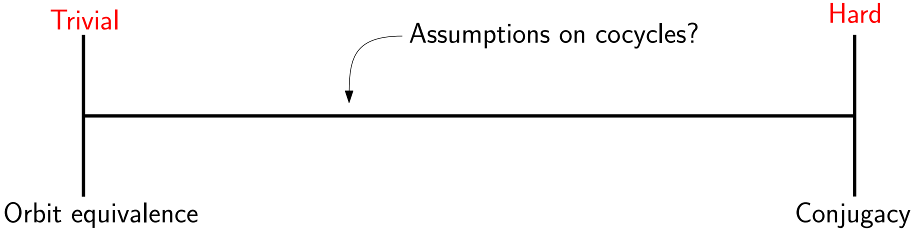
Example with $\Psi = \mathrm{id}_X$:

$$Tx = S^{c_T(x)}(x) \text{ and } Sx = T^{c_S(x)}(x)$$




S
$c_S(x) = 2$
T
T
T
T
T
$T^{-1}x$
x
Tx
T^2x
$c_S(T^2x) = -3$
S


Trivial
Hard
Assumptions on cocycles?
Orbit equivalence
Conjugacy

Let $T, S \in \mathrm{Aut}(X, \mu)$ be ergodic transformations, let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$.

Definition (Carderi–Joseph–Le Maître–Tessera 2023)

T and S are **φ-OE** if they admit an orbit equivalence whose cocycles are φ-integrable:

- $\displaystyle\int_X \varphi(|c_T(x)|)\mathrm{d}\mu(x) < +\infty;$
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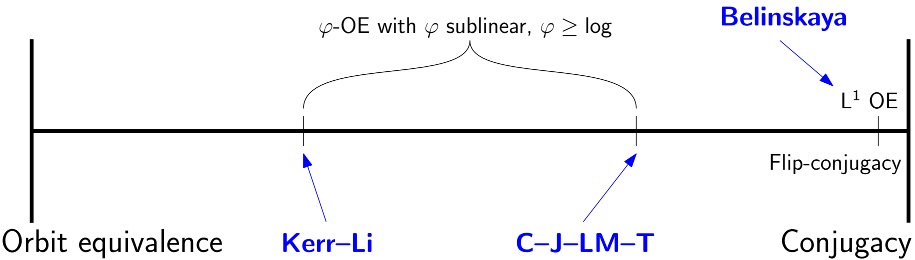
L^p OE $:=$ φ-OE with $\varphi(x) = x^p$ ($p < 1$ is allowed!)

L^∞ OE $\Rightarrow$ exp-OE $\Rightarrow$ L^p OE $\Rightarrow$ log-OE


φ-OE with φ sublinear, $\varphi \geq \log$
Belinskaya
L^1 OE
Flip-conjugacy
Orbit equivalence
Kerr–Li
C–J–LM–T
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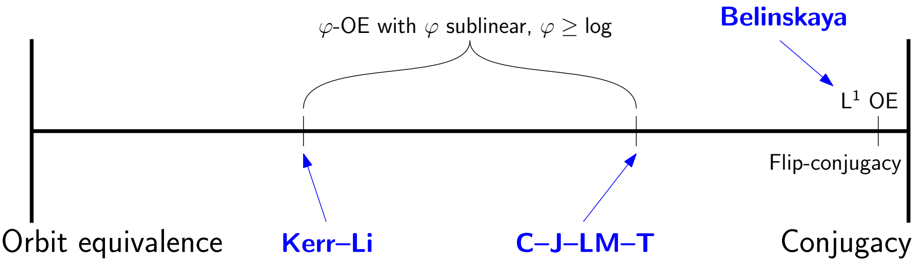
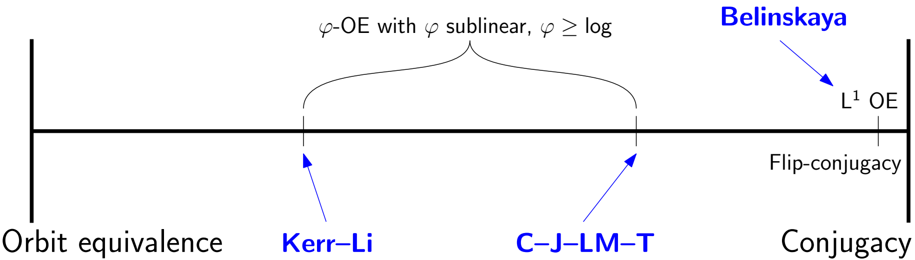


Theorem (Belinskaya 1969)

*If T and S are L^1 OE, then T is conjugate to S or S^{-1} (**flip-conjugacy**).*



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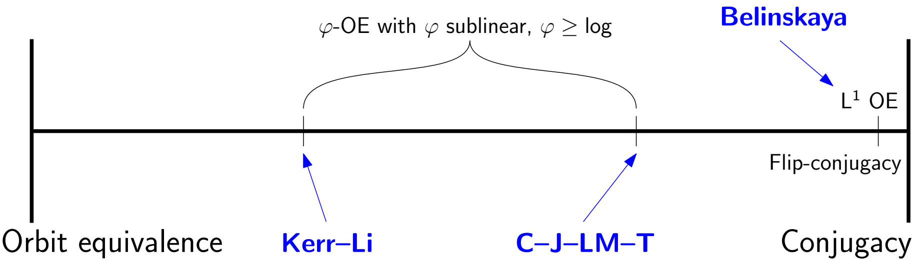

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Theorem (Kerr–Li 2023)

Entropy is preserved under log*-OE.*

Explicit constructions:

Theorem (Kerr–Li 2023, C. 2025)

Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(x) = o(x^{1/3})$.
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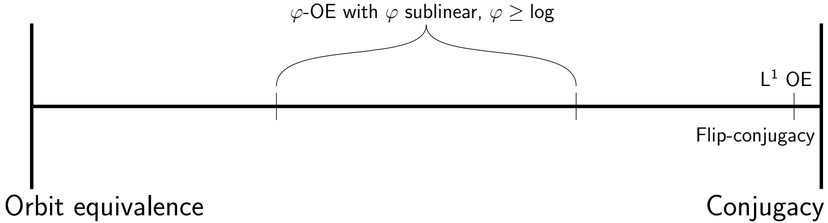
Question

What about quantitative orbit equivalence between Bernoulli shifts?



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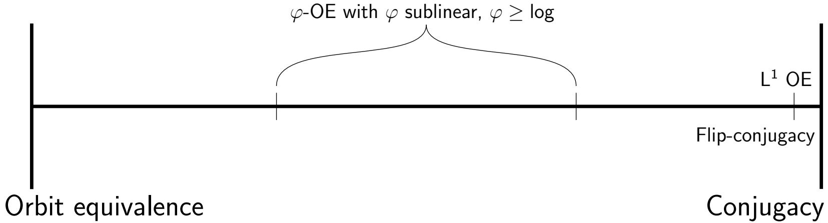

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- Same entropy $\overset{?}{\iff}$ log-OE
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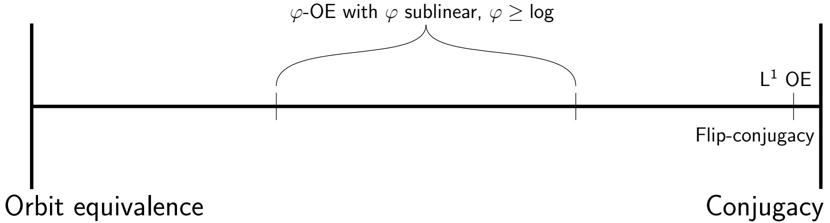

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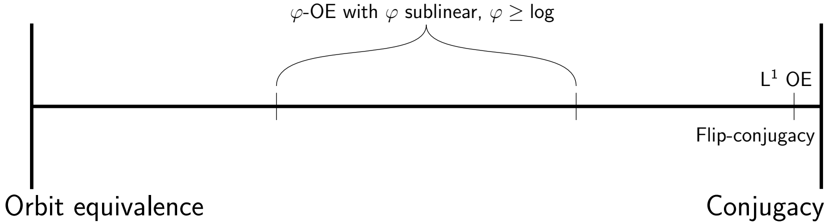

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For instance: For $\beta < 1$, is $\log^\beta$-OE trivial among ergodic transformations?
At least, we know that...

...Kerr and Li's theorem is optimal.

Theorem (C. 2026+)

Let S be an odometer with $k_{p_0} = \infty$ for some prime number p_0. Let $\alpha > 0$ or $\alpha = +\infty$. There exists $T \in \mathrm{Aut}(X, \mu)$ such that:

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Odometer S

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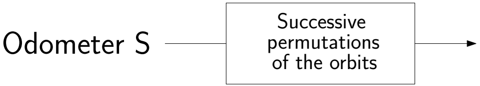

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Odometer S — $\boxed{\text{Successive permutations of the orbits}}$ → ☢ Odomutant T ☢

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Theorem (C. 2026+, weaker version)

Let S be the universal odometer. There exists $T \in \mathrm{Aut}(X, \mu)$ such that:

- $\mathrm{h}_\mu(T) > 0$;
- *T and S are $\log^\beta$-OE for all $\beta < 1$.*

Another definition of odometer.

$X = \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$

Combinatorial structure:

Definition (n-cylinders)

Given $i_0 \in \{0, 1, \dots, q_0 - 1\}, \dots, i_{n-1} \in \{0, 1, \dots, q_{n-1} - 1\}$,

$$[i_0, \dots, i_{n-1}]_n \coloneqq \{(x_i)_{n \geq 0} \in X \mid x_0 = i_0, \dots, x_{n-1} = i_{n-1}\}.$$

$q_0 = 4$

$[3]_1$	
$[2]_1$	
$[1]_1$	
$[0]_1$	

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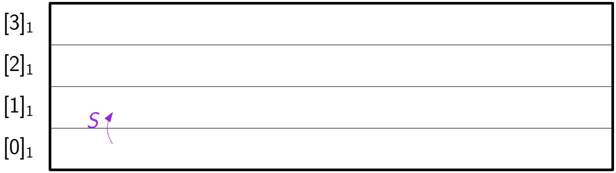

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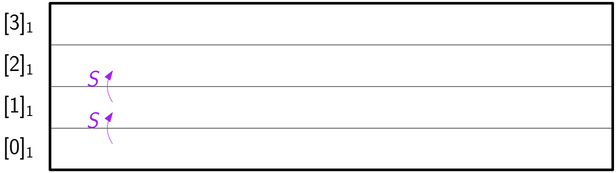

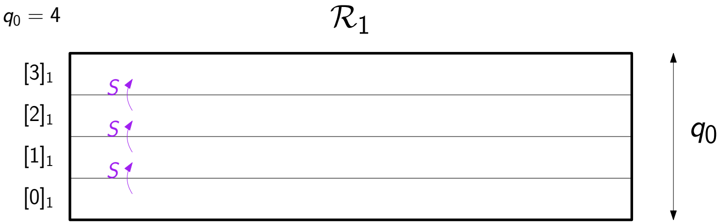
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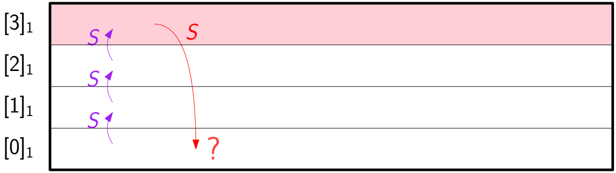
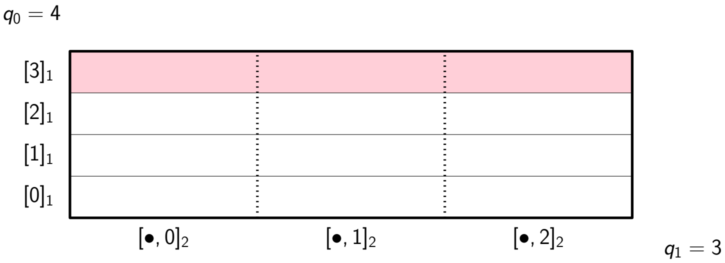
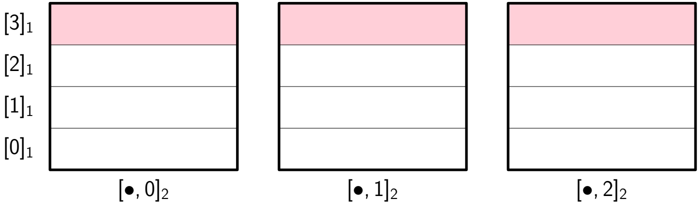
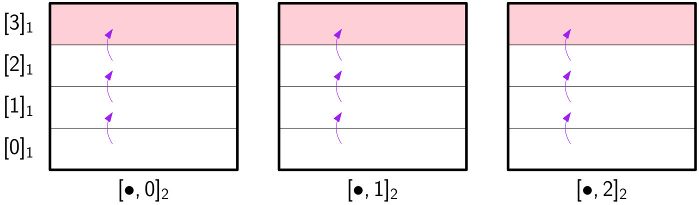
S
S
S
q_0

$q_0 = 4$

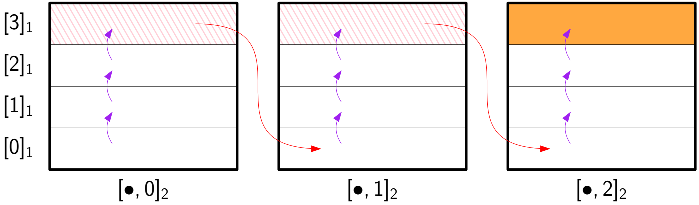


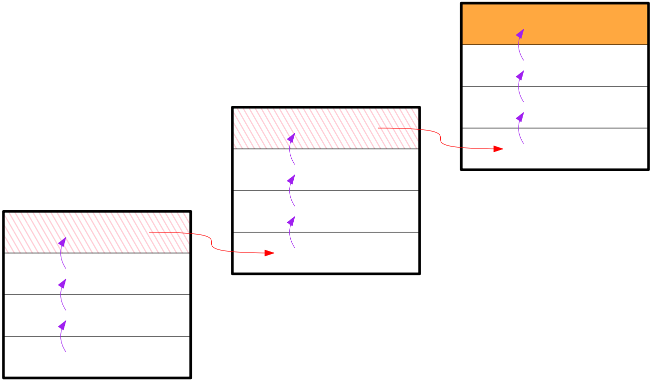

$q_0 = 4$
$[3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, 2]_2$
$q_1 = 3$

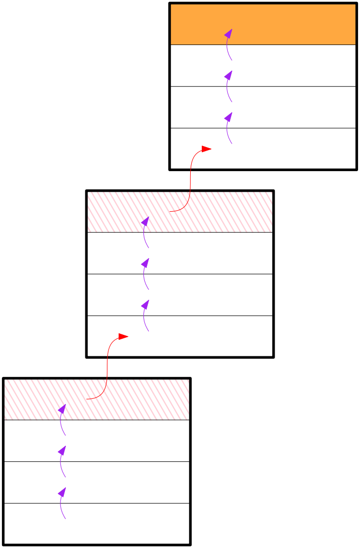

[3]1
[2]1
[1]1
[0]1
[•, 0]2
[•, 1]2
[•, 2]2

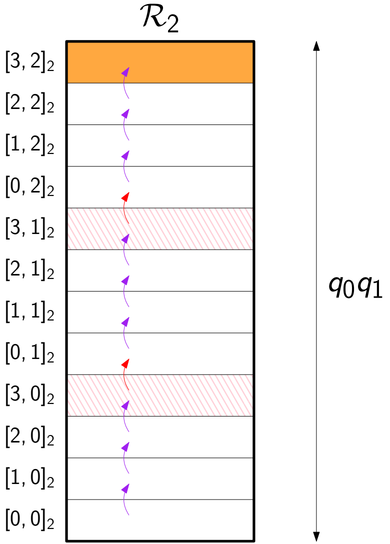


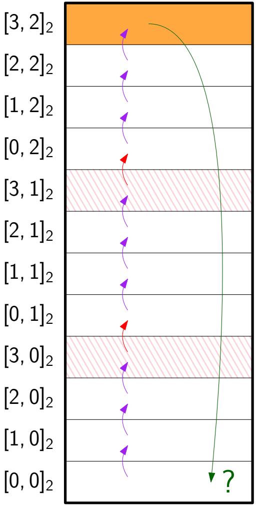
$[3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, 2]_2$

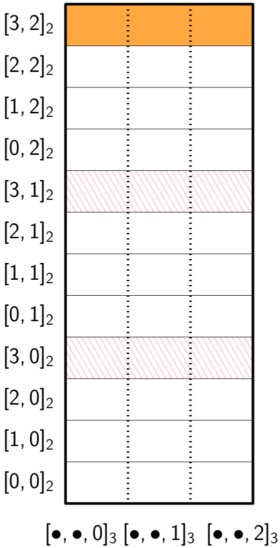

$[3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, 2]_2$



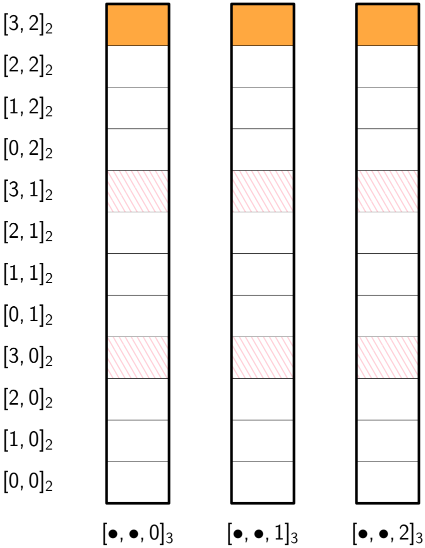


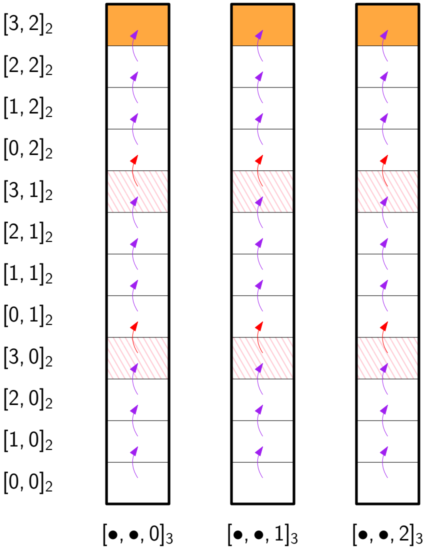

$\mathcal{R}_2$
$[3,2]_2$
$[2,2]_2$
$[1,2]_2$
$[0,2]_2$
$[3,1]_2$
$[2,1]_2$
$[1,1]_2$
$[0,1]_2$
$[3,0]_2$
$[2,0]_2$
$[1,0]_2$
$[0,0]_2$
q_0q_1

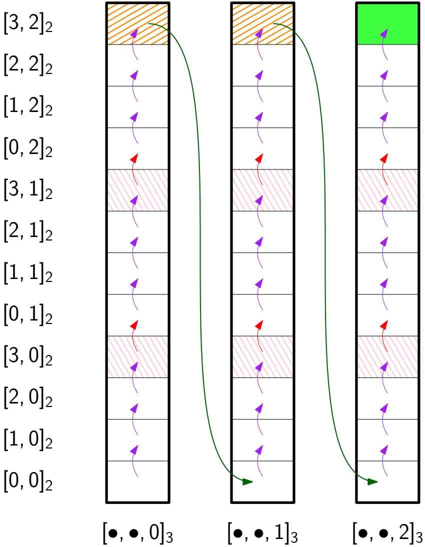

[3, 2]2
[2, 2]2
[1, 2]2
[0, 2]2
[3, 1]2
[2, 1]2
[1, 1]2
[0, 1]2
[3, 0]2
[2, 0]2
[1, 0]2
[0, 0]2
?


$[3, 2]_2$
$[2, 2]_2$
$[1, 2]_2$
$[0, 2]_2$
$[3, 1]_2$
$[2, 1]_2$
$[1, 1]_2$
$[0, 1]_2$
$[3, 0]_2$
$[2, 0]_2$
$[1, 0]_2$
$[0, 0]_2$
$[\bullet, \bullet, 0]_3$ $[\bullet, \bullet, 1]_3$ $[\bullet, \bullet, 2]_3$

$q_2 = 3$


$[3,2]_2$
$[2,2]_2$
$[1,2]_2$
$[0,2]_2$
$[3,1]_2$
$[2,1]_2$
$[1,1]_2$
$[0,1]_2$
$[3,0]_2$
$[2,0]_2$
$[1,0]_2$
$[0,0]_2$
$[\bullet,\bullet,0]_3$
$[\bullet,\bullet,1]_3$
$[\bullet,\bullet,2]_3$


[3, 2]2
[2, 2]2
[1, 2]2
[0, 2]2
[3, 1]2
[2, 1]2
[1, 1]2
[0, 1]2
[3, 0]2
[2, 0]2
[1, 0]2
[0, 0]2
[•, •, 0]3
[•, •, 1]3
[•, •, 2]3


$[3,2]_2$
$[2,2]_2$
$[1,2]_2$
$[0,2]_2$
$[3,1]_2$
$[2,1]_2$
$[1,1]_2$
$[0,1]_2$
$[3,0]_2$
$[2,0]_2$
$[1,0]_2$
$[0,0]_2$
$[\bullet,\bullet,0]_3$
$[\bullet,\bullet,1]_3$
$[\bullet,\bullet,2]_3$

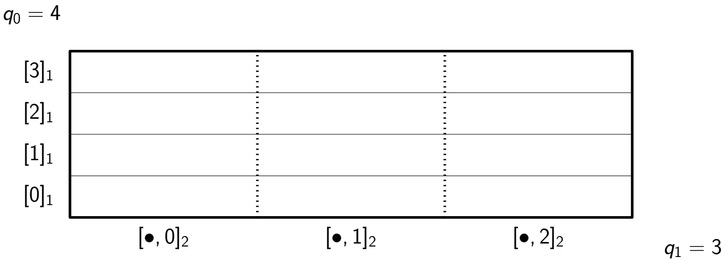
Odometer:

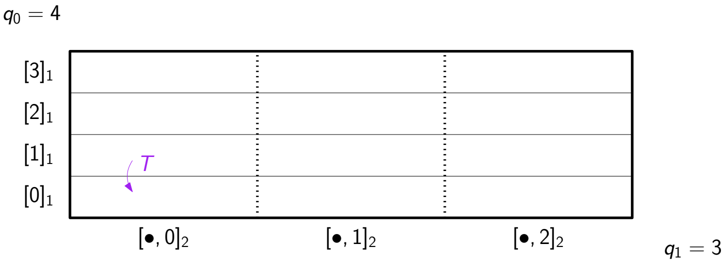
$$
\begin{tikzcd}
 & {[1]_1} \arrow[r, bend left, "S"] & {[2]_1} \arrow[dr, bend left, dash] & \\
{[0]_1} \arrow[ur, bend left, "S"] & & & \vdots \arrow[dl, bend left, "S"] \\
 & {[q_0 - 1]_1} \arrow[ul, bend left, "S"] & {[q_0 - 2]_1} \arrow[l, bend left, "S"] &
\end{tikzcd}
$$

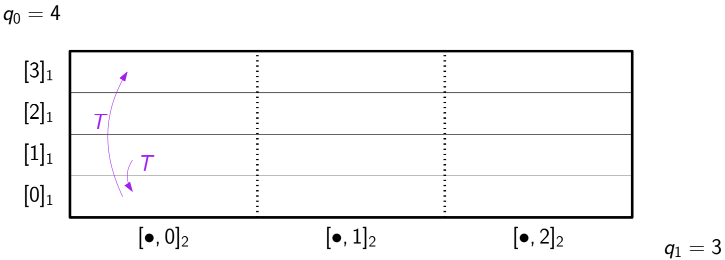
With an odomutant: less "predictable" dynamics

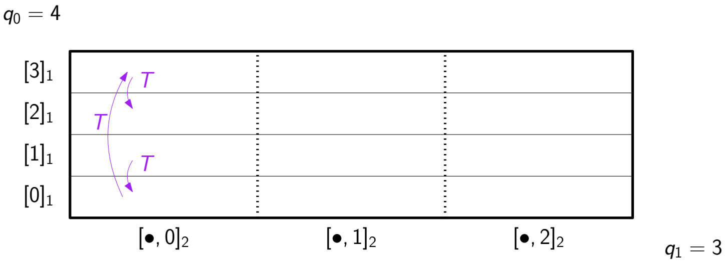
$q_0 = 4$

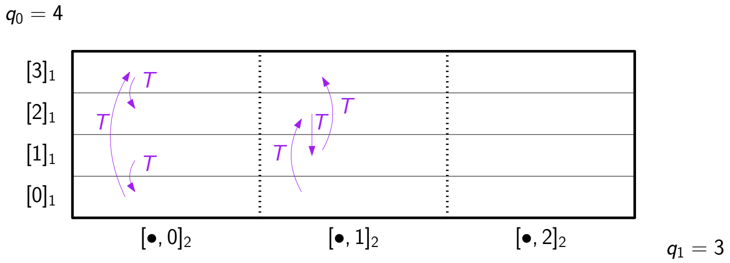
$[3]_1$	
$[2]_1$	
$[1]_1$	
$[0]_1$	

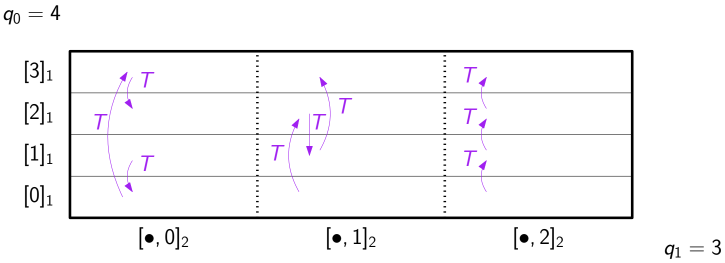

$q_0 = 4$
$[3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, 2]_2$
$q_1 = 3$


$q_0 = 4$
$[3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
T
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, 2]_2$
$q_1 = 3$

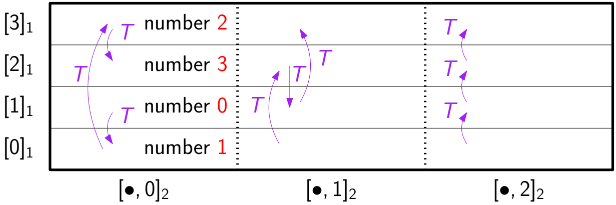

$q_0 = 4$
$[3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
T
T
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, 2]_2$
$q_1 = 3$


$q_0 = 4$
$[3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
T
T
T
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, 2]_2$
$q_1 = 3$


$q_0 = 4$
$[3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
T
T
T
T
T
T
T
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, 2]_2$
$q_1 = 3$


$q_0 = 4$
$[3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
T
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, 2]_2$
$q_1 = 3$

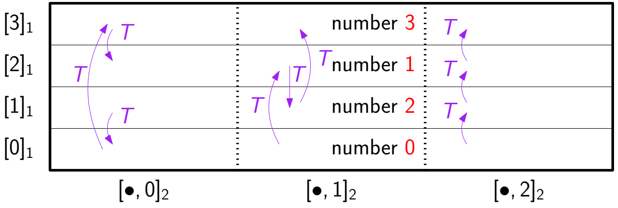
$q_0 = 4$



$q_1 = 3$

Permutations :
$0 \mapsto 1$
$1 \mapsto 0$
$2 \mapsto 3$
$3 \mapsto 2$

$q_0 = 4$

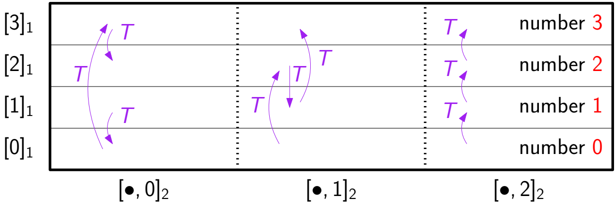


$q_1 = 3$

Permutations :

$[\bullet, 0]_2$	$[\bullet, 1]_2$
$0 \mapsto 1$	$0 \mapsto 0$
$1 \mapsto 0$	$1 \mapsto 2$
$2 \mapsto 3$	$2 \mapsto 1$
$3 \mapsto 2$	$3 \mapsto 3$

$q_0 = 4$



$q_1 = 3$

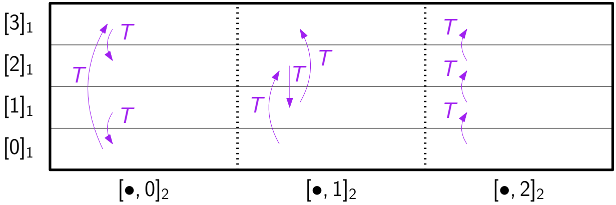
Permutations :

$[\bullet, 0]_2$	$[\bullet, 1]_2$	$[\bullet, 2]_2$
$0 \mapsto 1$	$0 \mapsto 0$	$0 \mapsto 0$
$1 \mapsto 0$	$1 \mapsto 2$	$1 \mapsto 1$
$2 \mapsto 3$	$2 \mapsto 1$	$2 \mapsto 2$
$3 \mapsto 2$	$3 \mapsto 3$	$3 \mapsto 3$

$\sigma^{(0)}_{x_1}$ → step 0

$\sigma^{(0)}_{x_1}$ → subcolumn $[\bullet, x_1]_2$

$q_0 = 4$

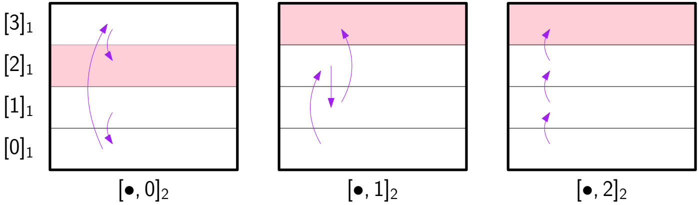


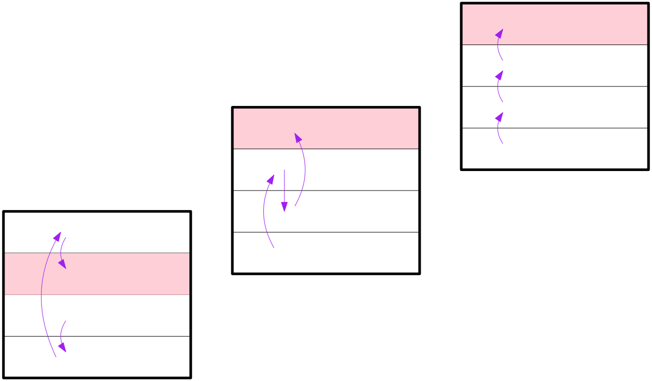
$q_1 = 3$

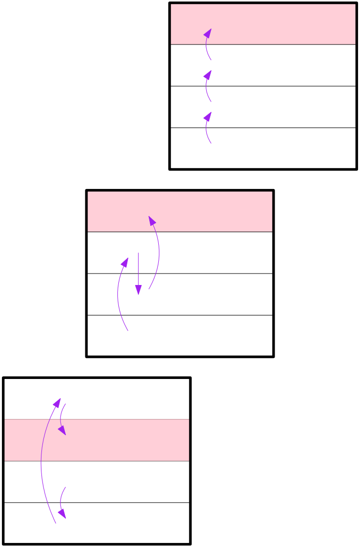
Permutations : $\sigma^{(0)}_0$: $0 \mapsto 1$, $1 \mapsto 0$, $2 \mapsto 3$, $3 \mapsto 2$

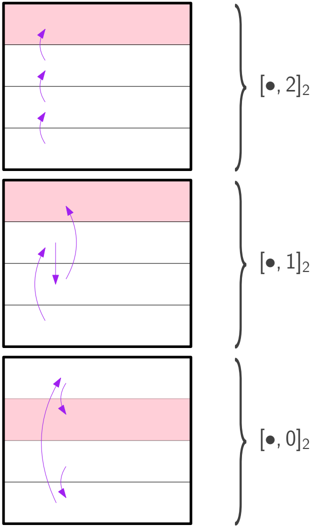
$\sigma^{(0)}_1$: $0 \mapsto 0$, $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$

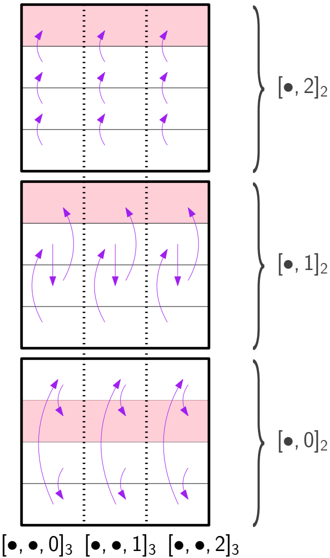
$\sigma^{(0)}_2$: $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 3$

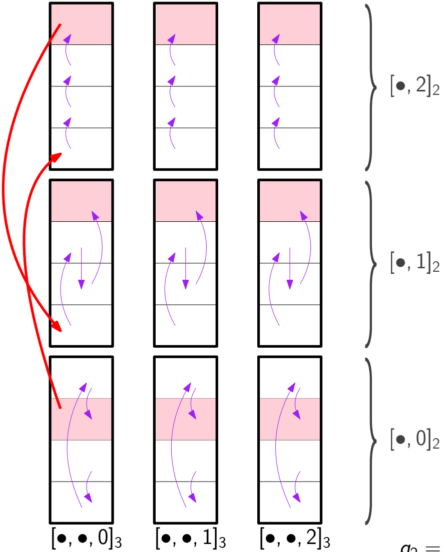

$[3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, 2]_2$



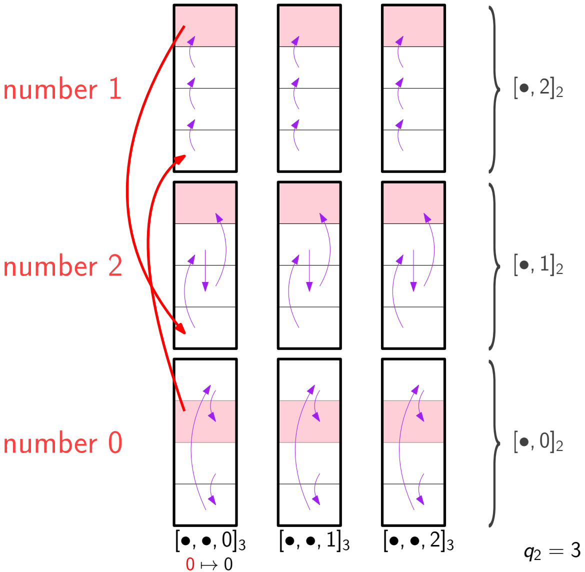



$[\bullet, 2]_2$
$[\bullet, 1]_2$
$[\bullet, 0]_2$

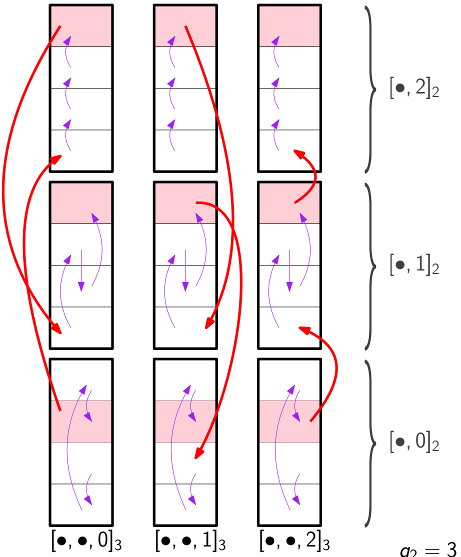

[●, 2]$_2$
[●, 1]$_2$
[●, 0]$_2$
[●, ●, 0]$_3$
[●, ●, 1]$_3$
[●, ●, 2]$_3$



$q_2 = 3$

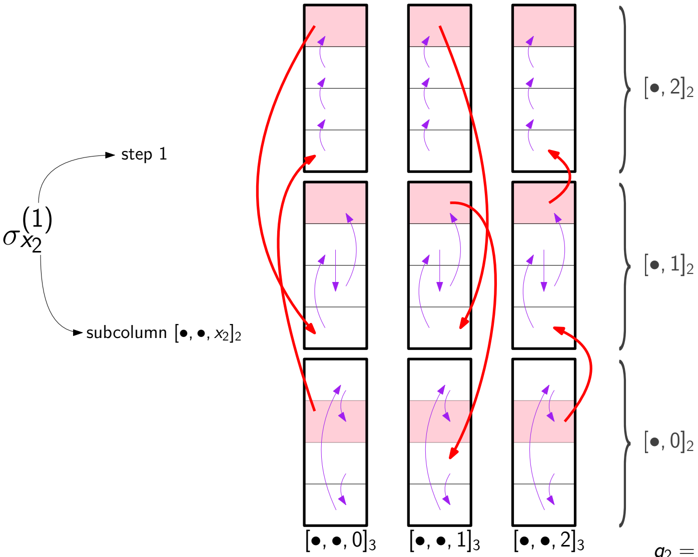


Permutations : $0 \mapsto 0$, $1 \mapsto 2$, $2 \mapsto 1$



Permutations :

$[\bullet, \bullet, 0]_3$	$[\bullet, \bullet, 1]_3$	$[\bullet, \bullet, 2]_3$
$0 \mapsto 0$	$0 \mapsto 2$	$0 \mapsto 0$
$1 \mapsto 2$	$1 \mapsto 1$	$1 \mapsto 1$
$2 \mapsto 1$	$2 \mapsto 0$	$2 \mapsto 2$



$q_2 = 3$

Permutations :
$$\sigma_0^{(1)} : \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 2 \\ 2 \mapsto 1 \end{array} \qquad \sigma_1^{(1)} : \begin{array}{l} 0 \mapsto 2 \\ 1 \mapsto 1 \\ 2 \mapsto 0 \end{array} \qquad \sigma_2^{(1)} : \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 1 \\ 2 \mapsto 2 \end{array}$$

An odomutant T is the data of

- **integers $\boldsymbol{q_0, q_1, q_2} \ldots \geq 2$**
  and the odometer S on $X = \prod_{n \geq 0} \{0, 1, \ldots, q_n - 1\}$;

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- for every $n \geq 0$:
  q_{n+1} **permutations** of $\{0, 1, \ldots, q_n - 1\}$: $\boldsymbol{\sigma_0^{(n)}, \sigma_1^{(n)}, \ldots, \sigma_{q_{n+1}-1}^{(n)}}$.

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At step n:

x_{n+1} $(n+1)$-th coordinate $\in \{0, \ldots, q_{n+1} - 1\}$
represents the block $[\bullet, \ldots, \bullet, \bullet, x_{n+1}]_{n+2}$

$\downarrow$

$\sigma_{x_{n+1}}^{(n)}$ permutation on the n-th coordinate $\in \{0, \ldots, q_n - 1\}$

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represents the block $[\bullet, \ldots, \bullet, \bullet, x_{n+1}]_{n+2}$

$\downarrow$

$\sigma_{x_{n+1}}^{(n)}$ permutation on the n-th coordinate $\in \{0, \ldots, q_n - 1\}$
$\rightarrow$ permutation of sub-blocks

$$[\bullet, \ldots, \bullet, 0, x_{n+1}]_{n+2},$$
$$[\bullet, \ldots, \bullet, 1, x_{n+1}]_{n+2},$$
$$\vdots$$
$$[\bullet, \ldots, \bullet, q_n, x_{n+1}]_{n+2}$$

$$\begin{array}{llll} \psi_n\colon & X & \longrightarrow & X \\ & (x_n)_{n\geq 0} & \longmapsto & \left(\sigma_{x_1}^{(0)}(x_0),\, \sigma_{x_2}^{(1)}(x_1), \dots,\, \sigma_{x_{n+1}}^{(n)}(x_n), x_{n+1}, x_{n+2}, \dots\right) \end{array}$$

$$\begin{array}{llll} \psi\colon & X & \longrightarrow & X \\ & (x_n)_{n\geq 0} & \longmapsto & \left(\sigma_{x_1}^{(0)}(x_0),\, \sigma_{x_2}^{(1)}(x_1),\, \sigma_{x_3}^{(2)}(x_2), \dots\right) = \lim_{n\to+\infty} \psi_n(x) \end{array}$$

$$\begin{array}{rccc} \psi_n\colon & X & \longrightarrow & X \\ & (x_n)_{n\geq 0} & \longmapsto & \left(\sigma_{x_1}^{(0)}(x_0),\ \sigma_{x_2}^{(1)}(x_1),\dots,\ \sigma_{x_{n+1}}^{(n)}(x_n), x_{n+1}, x_{n+2},\dots\right) \end{array}$$

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Definition (Odomutant T associated to the odometer S)

$T\colon \{x \in X \mid \psi(x) \neq (q_0 - 1, q_1 - 1, \dots)\} \longrightarrow \{x \in X \mid \psi(x) \neq (0, 0, \dots)\}$

$$\boldsymbol{Tx = \lim_{n\to+\infty} \psi_n^{-1} S \psi_n(x)}$$

$$
\begin{array}{rccl}
\psi_n\colon & X & \longrightarrow & X \\
& (x_n)_{n\geq 0} & \longmapsto & \left(\sigma_{x_1}^{(0)}(x_0),\, \sigma_{x_2}^{(1)}(x_1), \dots,\, \sigma_{x_{n+1}}^{(n)}(x_n), x_{n+1}, x_{n+2}, \dots\right)
\end{array}
$$

$$
\begin{array}{rccl}
\psi\colon & X & \longrightarrow & X \\
& (x_n)_{n\geq 0} & \longmapsto & \left(\sigma_{x_1}^{(0)}(x_0),\, \sigma_{x_2}^{(1)}(x_1),\, \sigma_{x_3}^{(2)}(x_2), \dots\right) = \lim_{n\to+\infty} \psi_n(x)
\end{array}
$$

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$$\boldsymbol{Tx = \lim_{n\to+\infty} \psi_n^{-1} S \psi_n(x)}$$

- $T \in \mathrm{Aut}(X, \mu)$;
- T and S have the same orbits (up to a null set);
- $\psi \circ T = S \circ \psi$ almost everywhere,
  so T factorizes onto S.

Proof of:

Theorem (C. 2026+, weaker version)

Let S be the universal odometer. There exists $T \in \mathrm{Aut}(X, \mu)$ such that:

- $\mathrm{h}_\mu(T) > 0$;
- *T and S are* $\log^\beta$*-OE for all $\beta < 1$.*

Calculation of $\mathrm{h}_\mu(T)$:

Proof of:

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Calculation of $\mathrm{h}_\mu(T)$:

- easier to compute topological entropy
  $\to$ $T \colon X \to X$ has to be continuous X

Proof of:

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Let S be the universal odometer. There exists $T \in \mathrm{Aut}(X, \mu)$ such that:

- $\mathrm{h}_\mu(T) > 0$;
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Calculation of $\mathrm{h}_\mu(T)$:

- easier to compute topological entropy
  $\to$ $T\colon X \to X$ has to be continuous X
- apply the variational principle
  $\to$ If T is uniquely ergodic, then $\mathrm{h}_{\mathrm{top}}(T) = \mathrm{h}_\mu(T)$

HYPOTHESIS 1:

For every $n \geq 0$,

0 et $q_n - 1$ are fixed point of the permutations $\sigma_0^{(n)}, \sigma_1^{(n)}, \ldots, \sigma_{q_{n+1}-1}^{(n)}$.

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$\{x \in X \mid \psi(x) = (q_0 - 1, q_1 - 1, \ldots)\} = \{(q_0 - 1, q_1 - 1, \ldots)\}$
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Property

- *$T\colon X \to X$ is a homeomorphism (so is continuous).*
- *For every $x \in X$ (not only "up to a null set"),*
  $\mathrm{Orb}_T(x) = \mathrm{Orb}_S(x)$ *(so T is uniquely ergodic).*

Goal:

- find the parameters:
  - the integers $\boldsymbol{q_n}$ (and the underlying odometer S);
  - and the permutations $\boldsymbol{\sigma_i^{(n)}}$ (satisfying HYPOTHESIS 1),

  such that $\mathrm{h}_{\mathrm{top}}(T) > 0$;

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  such that $\mathrm{h_{top}}(T) > 0$;
- ...in the hope of getting an orbit equivalence which is $\log^{\beta}$-integrable for every $\beta < 1$.

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Reminder

For $i \in \{0, 1, \ldots, q_0 - 1\}$, $[i]_1 = \{(x_n)_{n\geq 0} \in X \mid x_0 = i\}$ *(1-cylinder)*

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$$\mathrm{h}_{\mathrm{top}}(T) \geq \mathrm{h}_{\mathrm{top}}(T, \mathcal{U}_\star) = \lim_{n \to +\infty} \frac{\log \left| \, \mathcal{U}_\star \vee T^{-1}\mathcal{U}_\star \vee \cdots \vee T^{-n}\mathcal{U}_\star \, \right|}{n}$$

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Remark

Given $x \in X$,

$$\begin{aligned} & x \in [i_0]_1 \cap T^{-1}([i_1]_1) \cap \ldots \cap T^{-(n-1)}([i_{n-1}]_1) \\ \iff & x \in [i_0]_1,\ \ Tx \in [i_1]_1,\ \ \ldots,\ \ T^{n-1}x = [i_{n-1}]_1 \end{aligned}$$

$\rightsquigarrow$ To x, we associate the n-word $(i_0, \ldots, i_{n-1})$

$$\mathrm{h}_{\mathrm{top}}(T) \geq \lim_{n\to+\infty} \frac{\log\left(\text{number of } n\text{-words } (i_0,\ldots,i_{n-1}) \text{ obtained from 1-cylinders}\right)}{n}$$

Reminder on HYPOTHESIS 1:
For every $n \geq 0$,
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For every $n \geq 0$,
$\sigma_0^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$ describe all the permutations of $\{0, 1, \dots, q_n - 1\}$ fixing 0 and $q_n - 1$, and are pairwise distinct.

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Proposition

With HYPOTHESIS 2, we produce a lot of words!

$$\mathrm{h_{top}}(T) \geq (\log q_0) - C \text{ where } C \text{ is a constant.}$$

If q_0 is large enough, then $\mathrm{h_{top}}(T) > 0$.

Proof of the proposition:

1. We prove that

$$\text{"number of } h_n\text{-words obtained from 1-cylinders"} \geq q_n$$

   where $h_{n+1} = q_0 q_1 \dots q_n$ is the height of the towers at step n.

   (Proof by drawings in the next two slides, cases $n = 0$ and $n = 1$)

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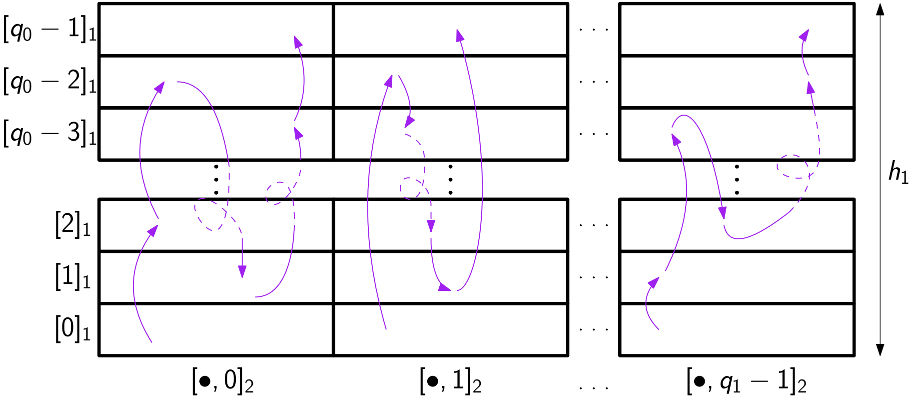
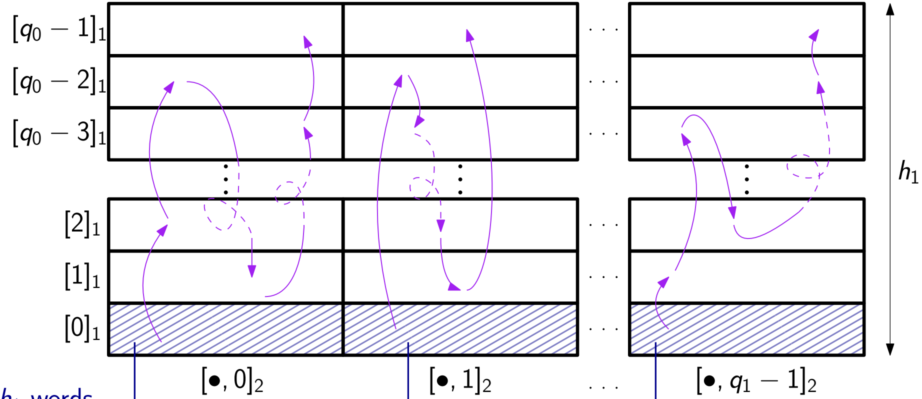
2. $\mathrm{h}_{\mathrm{top}}(T) \geq \lim_{n \to +\infty} \frac{\log(\text{number of } n\text{-words obtained from 1-cylinders})}{n}$,
   so $\mathrm{h}_{\mathrm{top}}(T) \geq \lim_{n \to +\infty} \frac{\log q_n}{h_n}$.

   To get $\frac{\log q_n}{h_n} \geq (\log q_0) - C$, we successively use
   - $q_{i+1} = (q_i - 2)!$;
   - $\log(q_i!) \geq q_i \log q_i$;
   - $h_{i+1} = q_i h_i$.



$[q_0 - 1]_1$
$[q_0 - 2]_1$
$[q_0 - 3]_1$
$[2]_1$
$[1]_1$
$[0]_1$
$[\bullet, 0]_2$
$[\bullet, 1]_2$
$[\bullet, q_1 - 1]_2$
h_1

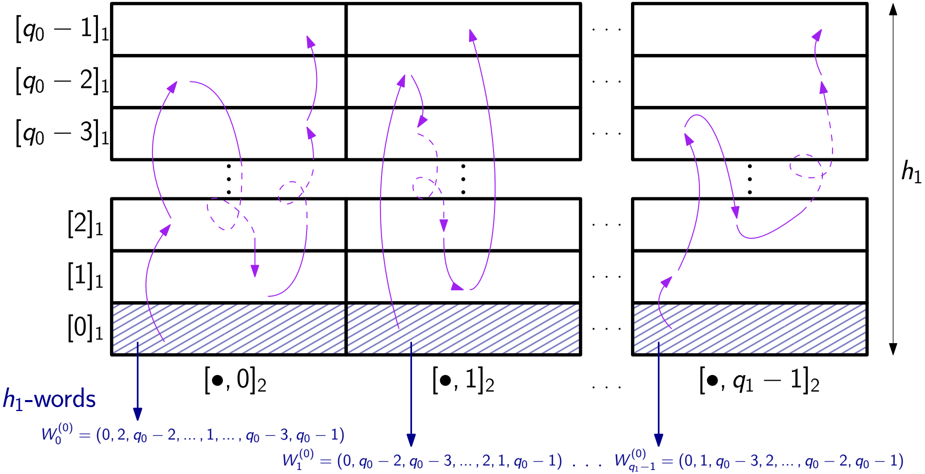
...



h_1-words

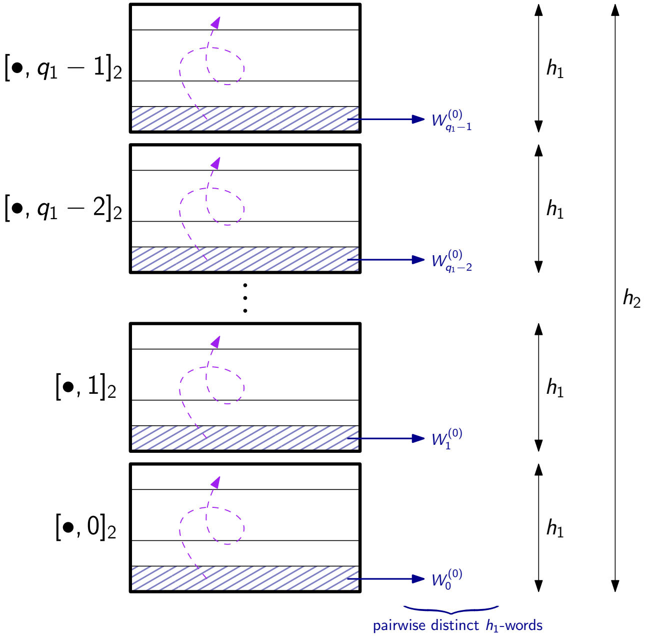
$W_0^{(0)} = (0, 2, q_0 - 2, \dots, 1, \dots, q_0 - 3, q_0 - 1)$

$W_1^{(0)} = (0, q_0 - 2, q_0 - 3, \dots, 2, 1, q_0 - 1)$ . . . $W_{q_1-1}^{(0)} = (0, 1, q_0 - 3, 2, \dots, q_0 - 2, q_0 - 1)$



$$\forall i \in \{0, \dots, q_1-1\}, \;\; W_i^{(0)} = (0, \sigma_i^{(0)}(1), \sigma_i^{(0)}(2), \dots, \sigma_i^{(0)}(q_0-2), q_0-1)$$

Pairwise distinct permutations $\Rightarrow$ at least q_1 h_1-words



$[\bullet, q_1 - 1]_2$
$W^{(0)}_{q_1-1}$
h_1
$[\bullet, q_1 - 2]_2$
$W^{(0)}_{q_1-2}$
h_1
h_2
$[\bullet, 1]_2$
$W^{(0)}_1$
h_1
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h_1
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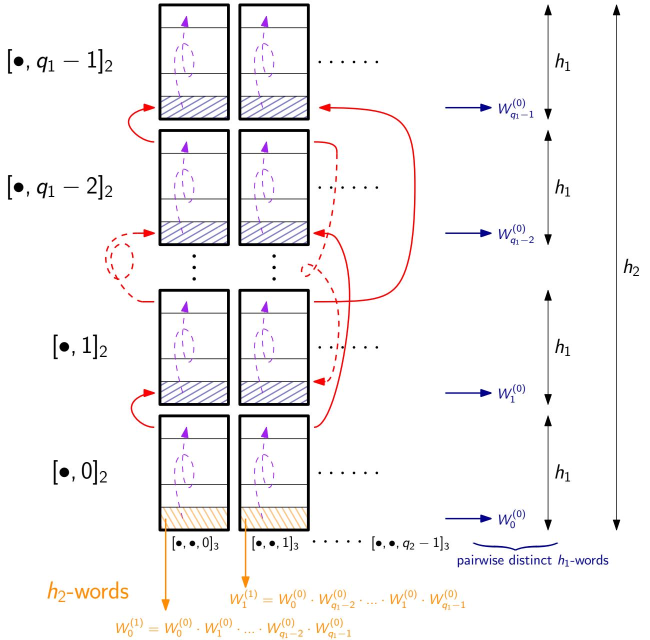

$[\bullet, q_1 - 1]_2$
$[\bullet, q_1 - 2]_2$
$[\bullet, 1]_2$
$[\bullet, 0]_2$
$W^{(0)}_{q_1-1}$
$W^{(0)}_{q_1-2}$
$W^{(0)}_{1}$
$W^{(0)}_{0}$
h_1
h_1
h_1
h_1
h_2
$[\bullet, \bullet, 0]_3$
$[\bullet, \bullet, 1]_3$
$[\bullet, \bullet, q_2 - 1]_3$
pairwise distinct h_1-words
h_2-words
$W^{(1)}_1 = W^{(0)}_0 \cdot W^{(0)}_{q_1-2} \cdot \ldots \cdot W^{(0)}_1 \cdot W^{(0)}_{q_1-1}$
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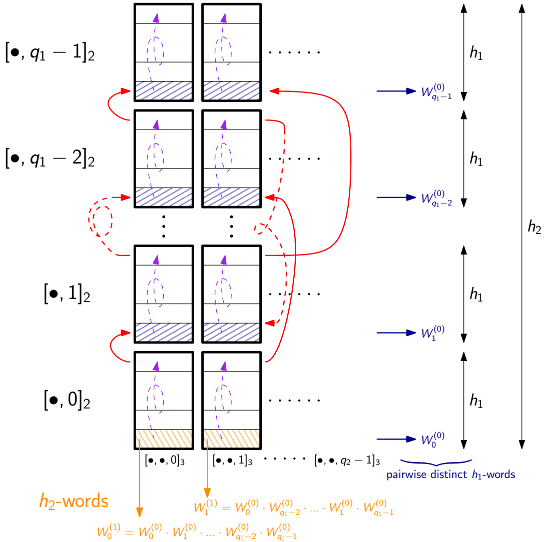

$$\forall i \in \{0, \ldots, q_2-1\},\ W^{(1)}_i = W^{(0)}_0 \cdot W^{(0)}_{\sigma^{(1)}_i(1)} \cdot W^{(0)}_{\sigma^{(1)}_i(2)} \cdot \ldots \cdot W^{(0)}_{\sigma^{(1)}_i(q_1-2)} \cdot W^{(0)}_{q_1-1}$$

Pairwise distinct permutations and pairwise distincts subwords $W^{(0)}_j$

$\Rightarrow$ at least q_2 h_2-words

Proposition

If $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is non-decreasing, then* $\displaystyle\int_X \varphi(|c_T(x)|)\mathrm{d}\mu(x)$ *and* $\displaystyle\int_X \varphi(|c_S(x)|)\mathrm{d}\mu(x)$ *are bounded above by* $\displaystyle\sum_{n\geq 0} \frac{\varphi(q_0 \dots q_{n+1})}{q_0 \dots q_n}$.

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Corollary

With $q_{n+1} = (q_n - 2)!$, the cocycles are $\log^\beta$-integrable for every $\beta < 1$.

This concludes the proof of the theorem on entropy (the weaker version).

Topological version:

Theorem (C. 2026+)

Let S be an odometer with $k_{p_0} = \infty$ for some prime number p_0. Let $\alpha > 0$ or $\alpha = +\infty$. There exists a Cantor minimal homeomorphism T such that:

- $\mathrm{h}_{\mathrm{top}}(T) = \alpha$;
- *T and S are* ***strongly orbit equivalent****, with $\log^{\beta}$-integrable cocycles for every $\beta < 1$.*

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Generalization of:

Theorem (Boyle–Handelman 1994)

Let S be the dyadic odometer, let $\alpha > 0$ or $\alpha = +\infty$. There exists a Cantor minimal homeomorphism T such that:

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Same construction, with different ways to establish the orbit equivalence:

- in Boyle and Handelman's proof: implicit way (complete invariant: dimension group);
- with odomutants: explicit construction.

Other topological interest:

Hypothesis 1 $\Rightarrow$ odomutant T is an almost 1-to-1 extension of the odometer S.

Hypothesis 1 & “pairwise different” in Hypothesis 2 $\Rightarrow$ T is a subshift

$\rightarrow$ We have built Toeplitz systems.