

# Odomutants and flexibility results for quantitative orbit equivalence

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Slides available here:

<https://corentincorreia.pages.math.cnrs.fr/Talks/beamerodomutant.pdf>

Optimality of Belinskaya's theorem for odometers:

Theorem (C. 2025+)

*Let  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a sublinear map and let  $S$  be an odometer. Then there exists  $T \in \text{Aut}(X, \mu)$  such that  $T$  and  $S$  are  $\varphi$ -OE but not flip-conjugate.*

Optimality of Kerr and Li's theorem:

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*Let  $S$  be the universal odometer, let  $\alpha > 0$  or  $\alpha = +\infty$ . There exists  $T \in \text{Aut}(X, \mu)$  such that:*

- $h_\mu(T) = \alpha$ ;
- $T$  and  $S$  are  $\log^\beta$ -OE for all  $\beta < 1$ .

Common construction:

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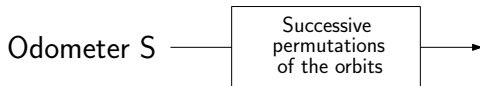
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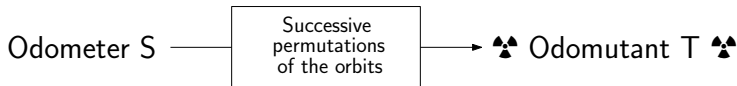
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Common construction:



Odometer  $S$



☢ Odomutant  $T$  ☢

- Let's define odomutants;
- prove a weaker version of the theorem on entropy:

Theorem (C. 2025+, weaker version)

Let  $S$  be the universal odometer. There exists  $T \in \text{Aut}(X, \mu)$  such that:

- $h_\mu(T) > 0$ ;
  - $T$  and  $S$  are  $\log^\beta$ -OE for all  $\beta < 1$ .
- and sketch the proof of the other theorem (optimality of Belinskaya's theorem for odometers)

Another definition of odometer.

### Combinatorial structure:

#### Definition ( $n$ -cylinders)

Given  $i_0 \in \{0, 1, \dots, q_0 - 1\}, \dots, i_{n-1} \in \{0, 1, \dots, q_{n-1} - 1\}$ ,

$$[i_0, \dots, i_{n-1}]_n := \{(x_i)_{n \geq 0} \in X \mid x_0 = i_0, \dots, x_{n-1} = i_{n-1}\}.$$

$$q_0 = 4$$

$[3]_1$

$[2]_1$

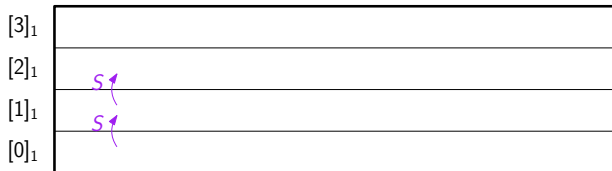
$[1]_1$

$[0]_1$

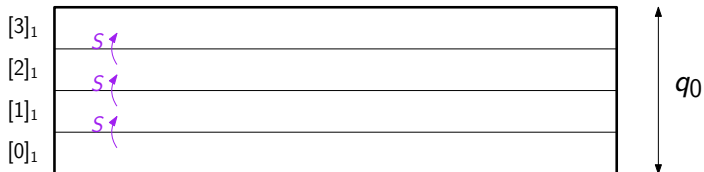

$$q_0 = 4$$

$[3]_1$	
$[2]_1$	
$[1]_1$	$s \uparrow$
$[0]_1$	

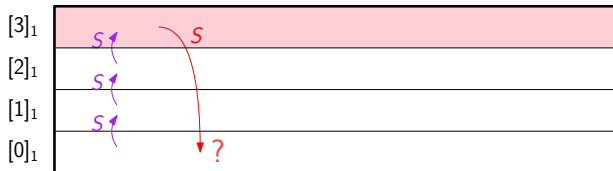
$$q_0 = 4$$



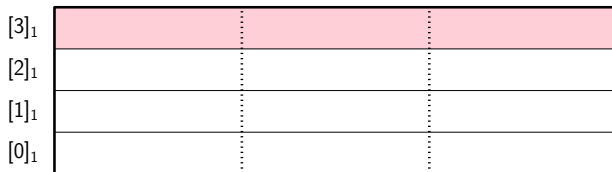
$$q_0 = 4$$

$$\mathcal{R}_1$$


$$q_0 = 4$$



$q_0 = 4$

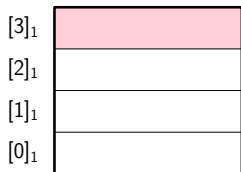


$[\bullet, 0]_2$

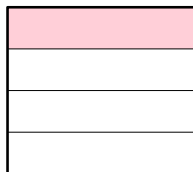
$[\bullet, 1]_2$

$[\bullet, 2]_2$

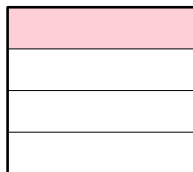
$q_1 = 3$



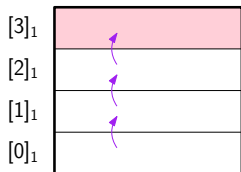
$[\bullet, 0]_2$



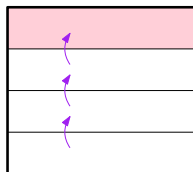
$[\bullet, 1]_2$



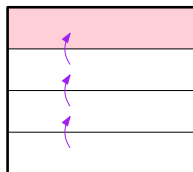
$[\bullet, 2]_2$



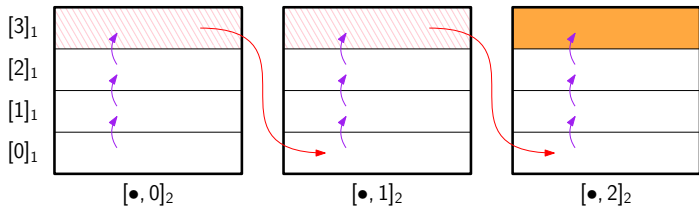
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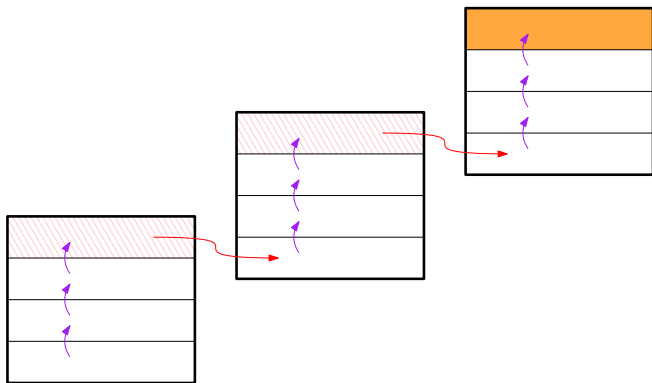


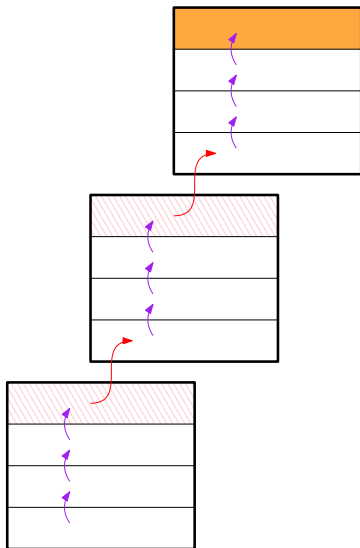
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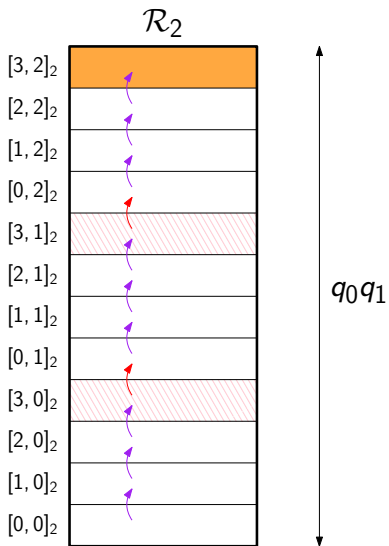


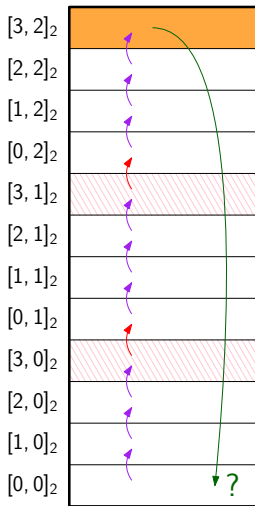
$[\bullet, 2]_2$

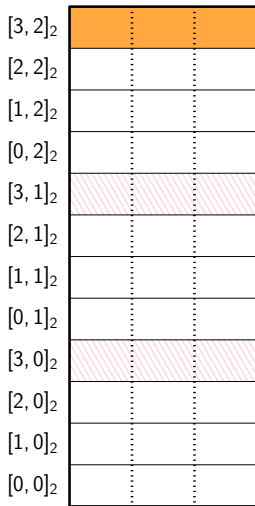






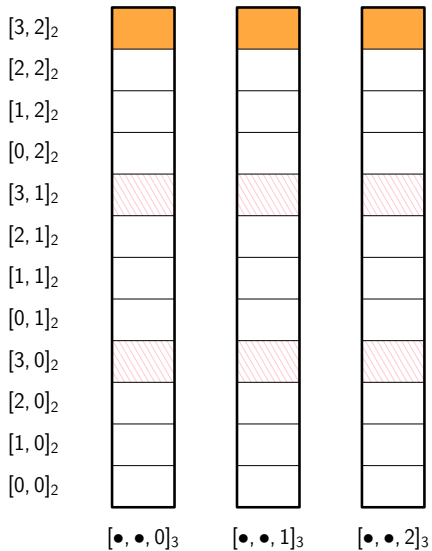


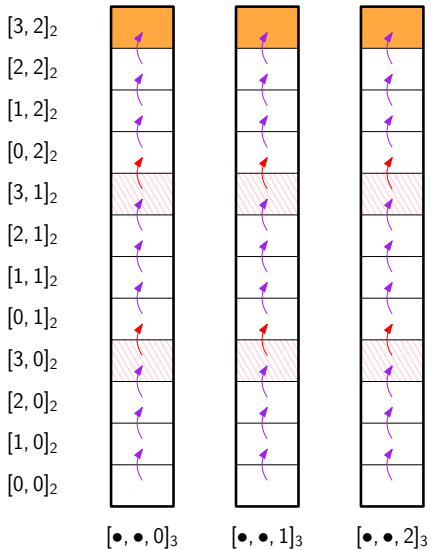


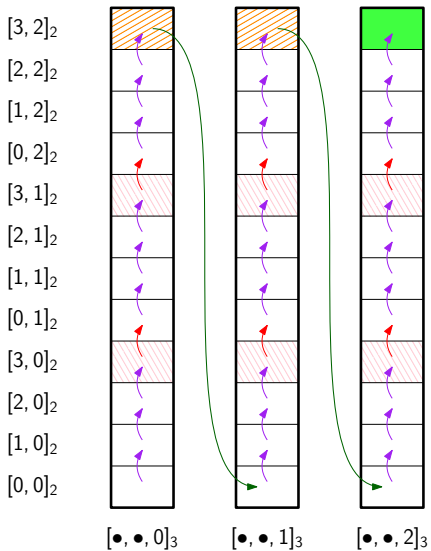


$[\bullet, \bullet, 0]_3$   $[\bullet, \bullet, 1]_3$   $[\bullet, \bullet, 2]_3$

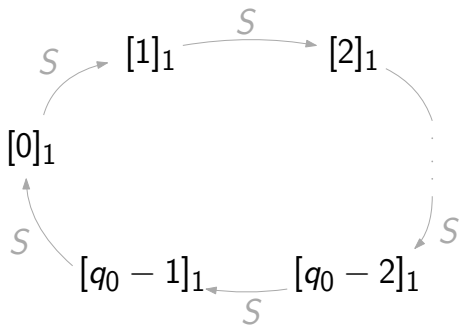
$$q_2 = 3$$







Odometer:



With an odomutant: less “predictable” dynamics

$$q_0 = 4$$

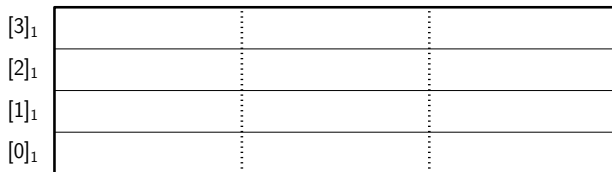
$[3]_1$

$[2]_1$

$[1]_1$

$[0]_1$


$q_0 = 4$



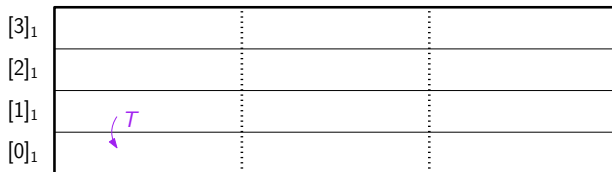
$[\bullet, 0]_2$

$[\bullet, 1]_2$

$[\bullet, 2]_2$

$q_1 = 3$

$q_0 = 4$



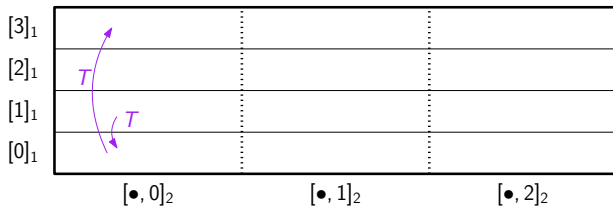
$[\bullet, 0]_2$

$[\bullet, 1]_2$

$[\bullet, 2]_2$

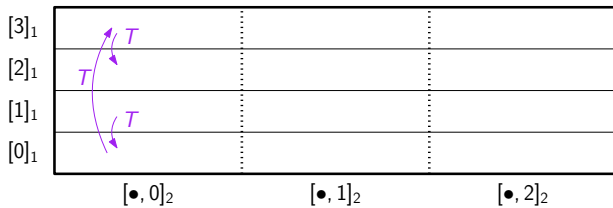
$q_1 = 3$

$q_0 = 4$



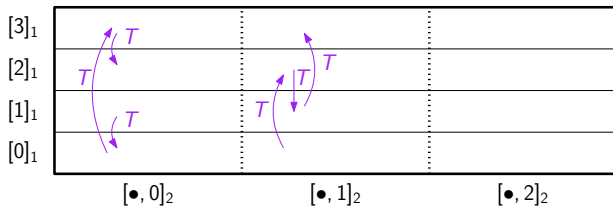
$q_1 = 3$

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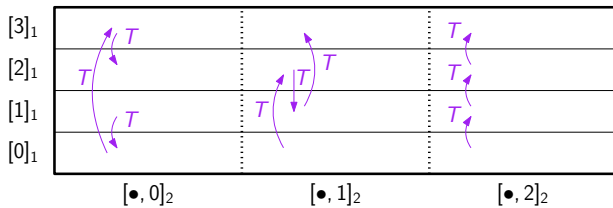
$q_1 = 3$

$q_0 = 4$



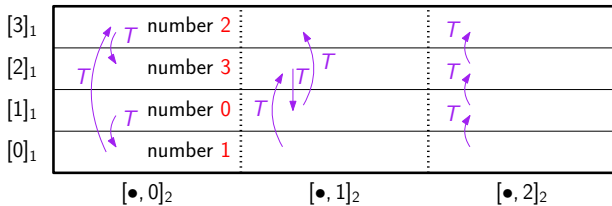
$q_1 = 3$

$q_0 = 4$



$q_1 = 3$

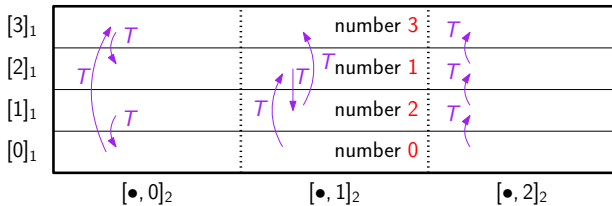
$q_0 = 4$



$q_1 = 3$

Permutations :  
 $0 \mapsto 1$   
 $1 \mapsto 0$   
 $2 \mapsto 3$   
 $3 \mapsto 2$

$q_0 = 4$

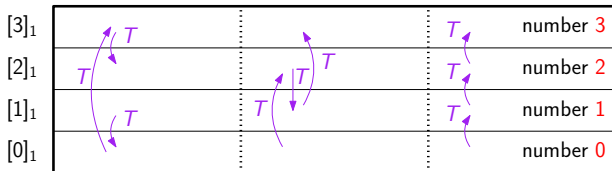


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 $1 \mapsto 0$   
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 $1 \mapsto 2$   
 $2 \mapsto 1$   
 $3 \mapsto 3$

$q_1 = 3$

$q_0 = 4$



$[\bullet, 0]_2$

$[\bullet, 1]_2$

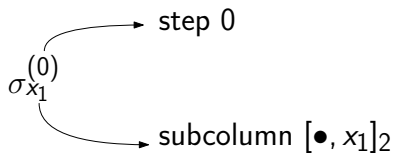
$[\bullet, 2]_2$

Permutations :  
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 $1 \mapsto 0$   
 $2 \mapsto 3$   
 $3 \mapsto 2$

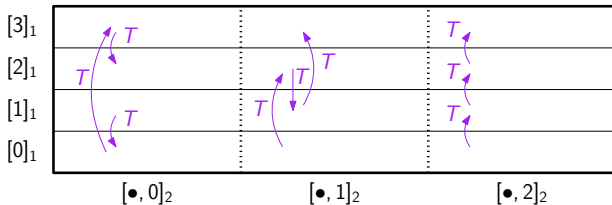
$0 \mapsto 0$   
 $1 \mapsto 2$   
 $2 \mapsto 1$   
 $3 \mapsto 3$

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 $1 \mapsto 1$   
 $2 \mapsto 2$   
 $3 \mapsto 3$

$q_1 = 3$



$q_0 = 4$

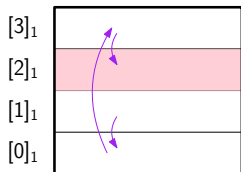


$q_1 = 3$

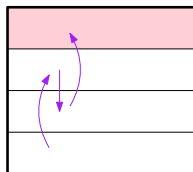
Permutations :  $\sigma_0^{(0)}$ :
   
 $0 \mapsto 1$ 
  
 $1 \mapsto 0$ 
  
 $2 \mapsto 3$ 
  
 $3 \mapsto 2$

$\sigma_1^{(0)}$ :
   
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 $1 \mapsto 2$ 
  
 $2 \mapsto 1$ 
  
 $3 \mapsto 3$

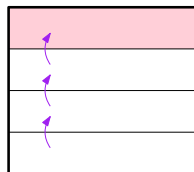
$\sigma_2^{(0)}$ :
   
 $0 \mapsto 0$ 
  
 $1 \mapsto 1$ 
  
 $2 \mapsto 2$ 
  
 $3 \mapsto 3$



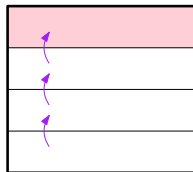
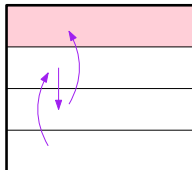
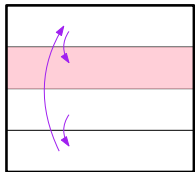
$[\bullet, 0]_2$

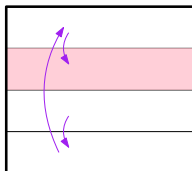
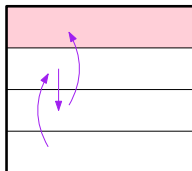
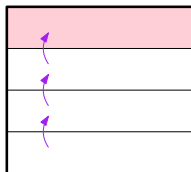


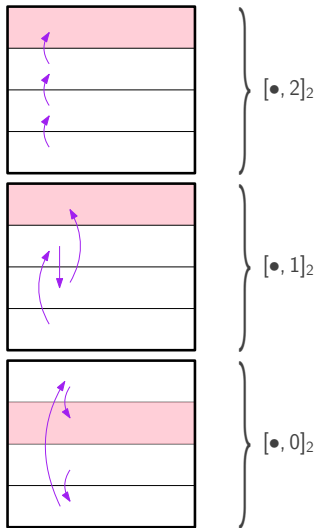
$[\bullet, 1]_2$

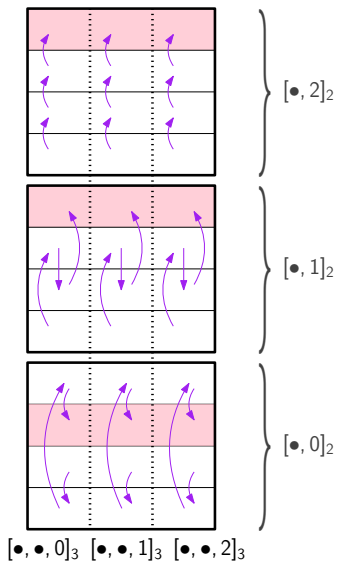


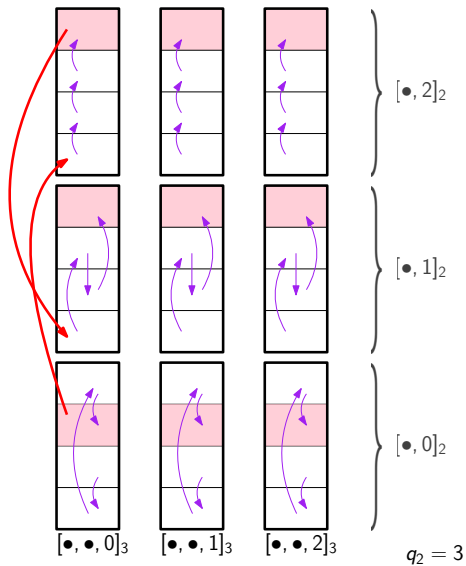
$[\bullet, 2]_2$







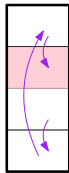
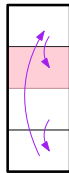
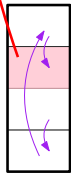
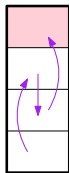
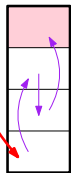
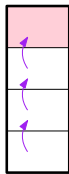
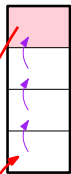




number 1

number 2

number 0



$[\bullet, \bullet, 0]_3$

$[\bullet, \bullet, 1]_3$

$[\bullet, \bullet, 2]_3$

$[\bullet, 2]_2$

$[\bullet, 1]_2$

$[\bullet, 0]_2$

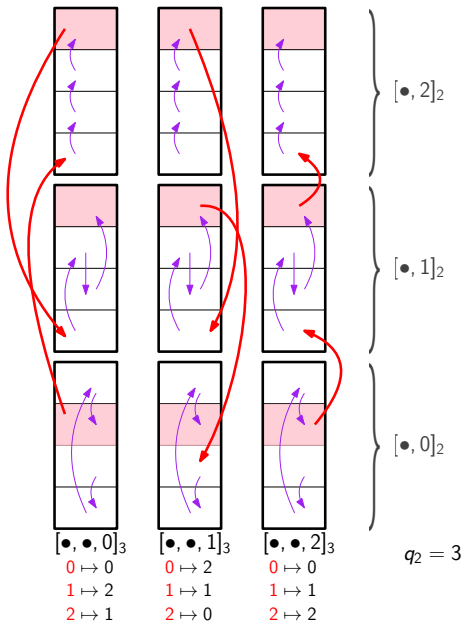
$q_2 = 3$

Permutations :

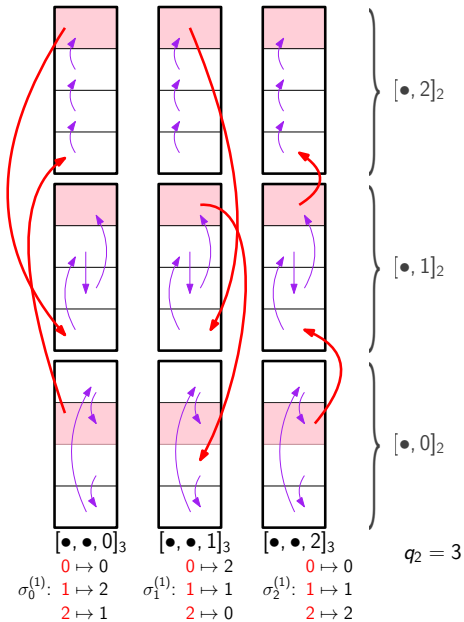
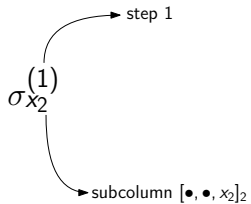
$0 \mapsto 0$

$1 \mapsto 2$

$2 \mapsto 1$



Permutations :



Permutations :

$$\sigma_0^{(1)}:$$

$$0 \mapsto 0$$

$$1 \mapsto 2$$

$$2 \mapsto 1$$

$$\sigma_1^{(1)}:$$

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$$1 \mapsto 1$$

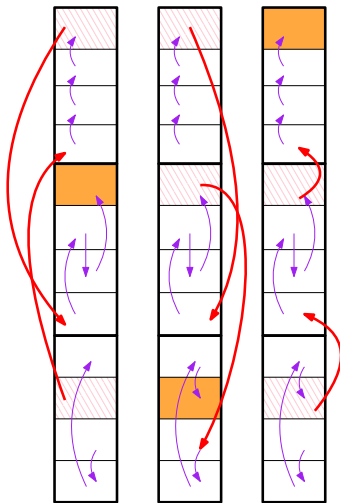
$$2 \mapsto 0$$

$$\sigma_2^{(1)}:$$

$$0 \mapsto 0$$

$$1 \mapsto 1$$

$$2 \mapsto 2$$



An odomutant  $T$  is the data of

- integers  $q_0, q_1, q_2 \dots \geq 2$

and the odometer  $S$  on  $X = \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$ ;

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- for every  $n \geq 0$ :  
 $q_{n+1}$  **permutations** of  $\{0, 1, \dots, q_n - 1\}$ :  $\sigma_0^{(n)}, \sigma_1^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$ .

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At step  $n$ :

$x_{n+1}$        $(n+1)$ -th coordinate  $\in \{0, \dots, q_{n+1} - 1\}$   
represents the block  $[\bullet, \dots, \bullet, \bullet, x_{n+1}]_{n+2}$

↓

$\sigma_{x_{n+1}}^{(n)}$       permutation on the  $n$ -th coordinate  $\in \{0, \dots, q_n - 1\}$

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represents the block  $[\bullet, \dots, \bullet, \bullet, x_{n+1}]_{n+2}$

↓

$\sigma_{x_{n+1}}^{(n)}$      permutation on the  **$n$ -th coordinate**  $\in \{0, \dots, q_n - 1\}$   
→ permutation of sub-blocks

$$\begin{aligned} & [\bullet, \dots, \bullet, 0, x_{n+1}]_{n+2}, \\ & [\bullet, \dots, \bullet, 1, x_{n+1}]_{n+2}, \\ & \vdots \\ & [\bullet, \dots, \bullet, q_n, x_{n+1}]_{n+2} \end{aligned}$$

$$\begin{aligned} \psi_n: \quad X &\longrightarrow X \\ (x_n)_{n \geq 0} &\longmapsto \left( \sigma_{x_1}^{(0)}(x_0), \sigma_{x_2}^{(1)}(x_1), \dots, \sigma_{x_{n+1}}^{(n)}(x_n), x_{n+1}, x_{n+2}, \dots \right) \end{aligned}$$

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Definition (Odomutant  $T$  associated to the odometer  $S$ )

$$T: \{x \in X \mid \psi(x) \neq (q_0 - 1, q_1 - 1, \dots)\} \longrightarrow \{x \in X \mid \psi(x) \neq (0, 0, \dots)\}$$

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- $T \in \text{Aut}(X, \mu)$ ;
- $T$  and  $S$  have the same orbits (up to a null set);
- $\psi \circ T = S \circ \psi$  almost everywhere,  
so  $T$  factorizes onto  $S$ .

Proof of:

Theorem (C. 2025+, weaker version)

Let  $S$  be the universal odometer. There exists  $T \in \text{Aut}(X, \mu)$  such that:

- $h_\mu(T) > 0$ ;
- $T$  and  $S$  are  $\log^\beta$ -OE for all  $\beta < 1$ .

Calculation of  $h_\mu(T)$ :

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Calculation of  $h_\mu(T)$ :

- easier to compute topological entropy  
→  $T: X \rightarrow X$  has to be continuous  $X$
- apply the variational principle  
→ If  $T$  is uniquely ergodic, then  $h_{\text{top}}(T) = h_\mu(T)$

## HYPOTHESIS 1:

For every  $n \geq 0$ ,  $0$  et  $q_n - 1$  are fixed point of the permutations  $\sigma_0^{(n)}, \sigma_1^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$ .

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### Property

- $T: X \rightarrow X$  is a homeomorphism (so is continuous).
- For every  $x \in X$  (not only "up to a null set"),  $\text{Orb}_T(x) = \text{Orb}_S(x)$  (so  $T$  is uniquely ergodic).

## Goal:

- find the parameters:
  - the integers  $q_n$  (and the underlying odometer  $S$ );
  - and the permutations  $\sigma_i^{(n)}$  (satisfying **HYPOTHESIS 1**),such that  $h_{\text{top}}(T) > 0$ ;

## Goal:

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- ...in the hope of getting an orbit equivalence which is  $\log^\beta$ -integrable for every  $\beta < 1$ .

## Topological entropy:

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$(\mathcal{U}_*)^n$  is the set of elements of the form

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$$h_{\text{top}}(T) \geq \lim_{n \rightarrow +\infty} \frac{\log(\text{number of } n\text{-words } (i_0, \dots, i_{n-1}) \text{ obtained from } 1\text{-cylinders})}{n}$$

**Reminder** on **HYPOTHESIS 1**:

For every  $n \geq 0$ ,  $0$  and  $q_n - 1$  are fixed points of the permutations

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**HYPOTHESIS 2**:

For every  $n \geq 0$ ,  $\sigma_0^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$  describe all the permutations of  $\{0, 1, \dots, q_n - 1\}$  fixing  $0$  and  $q_n - 1$ , and are pairwise distinct.

$$\leadsto q_{n+1} = (q_n - 2)!$$

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## Proposition

With **HYPOTHESIS 2**, we produce a lot of words !

$$h_{\text{top}}(T) \geq (\log q_0) - C \text{ where } C \text{ is a constant.}$$

If  $q_0$  is large enough, then  $h_{\text{top}}(T) > 0$ .

Proof of the proposition:

① We prove that

"number of  $h_n$ -words obtained from 1-cylinders"  $\geq q_n$

where  $h_{n+1} = q_0 q_1 \dots q_n$  is the height of the towers at step  $n$ .

(Proof by drawings in the next two slides, cases  $n = 0$  and  $n = 1$ )

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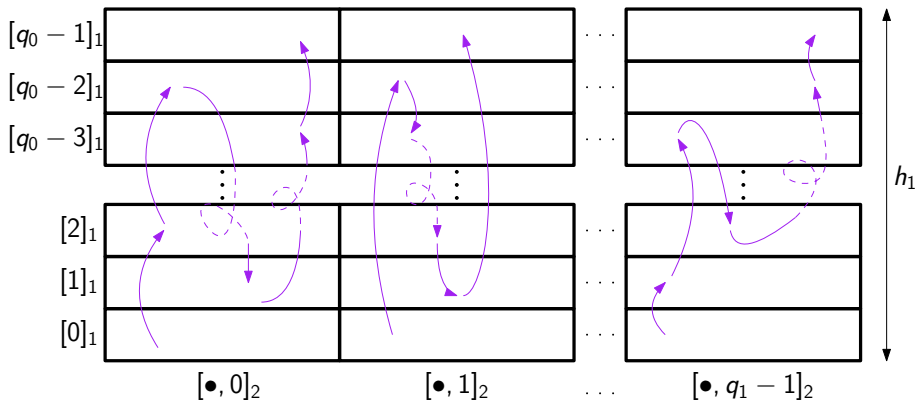
(Proof by drawings in the next two slides, cases  $n = 0$  and  $n = 1$ )

②  $h_{\text{top}}(T) \geq \lim_{n \rightarrow +\infty} \frac{\log(\text{number of } n\text{-words obtained from 1-cylinders})}{n}$ ,

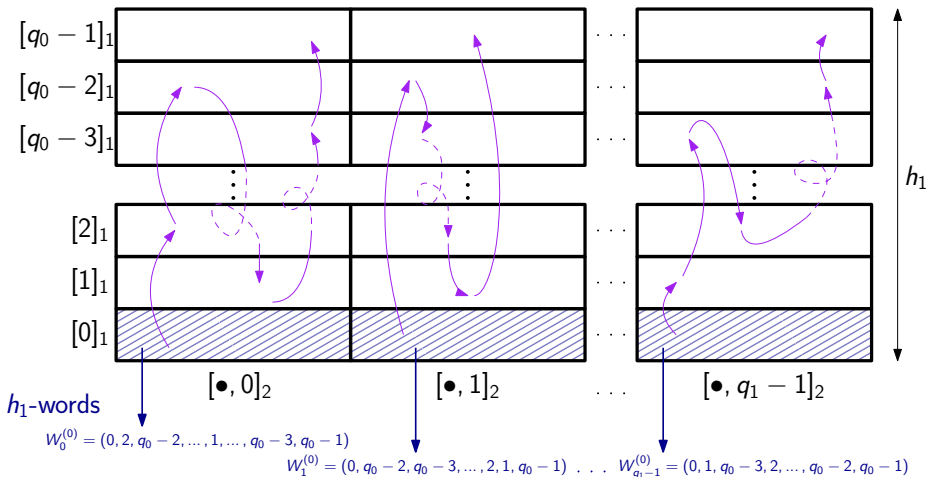
so  $h_{\text{top}}(T) \geq \lim_{n \rightarrow +\infty} \frac{\log q_n}{h_n}$ .

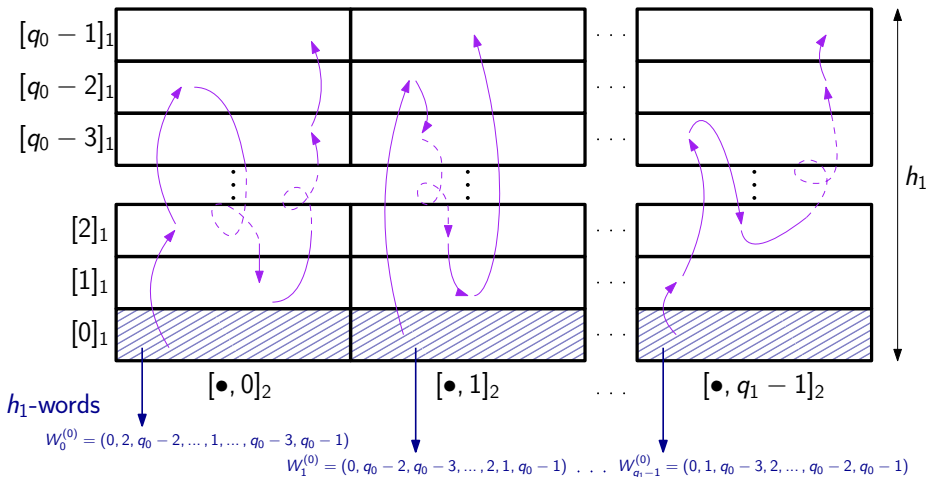
To get  $\frac{\log q_n}{h_n} \geq (\log q_0) - C$ , we successively use

- $q_{i+1} = (q_i - 2)!;$
- $\log(q_i!) \geq q_i \log q_i;$
- $h_{i+1} = q_i h_i.$



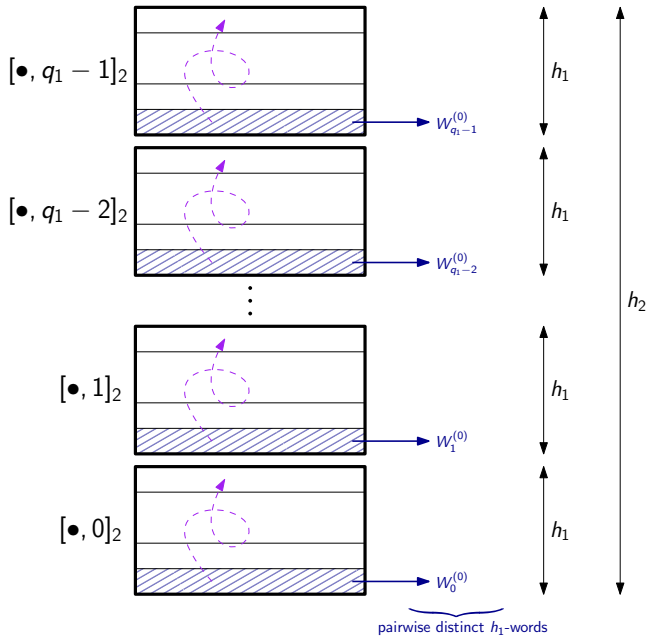
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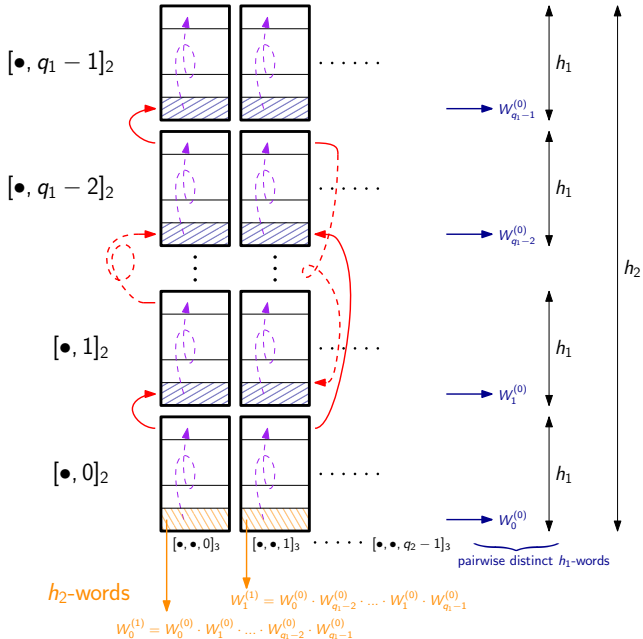


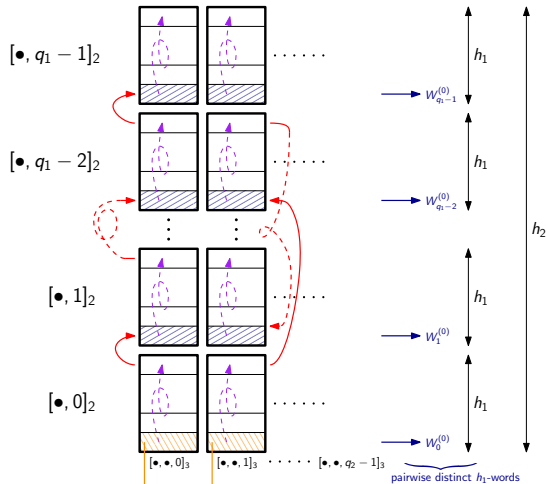


$$\forall i \in \{0, \dots, q_1 - 1\}, W_i^{(0)} = (0, \sigma_i^{(0)}(1), \sigma_i^{(0)}(2), \dots, \sigma_i^{(0)}(q_0 - 2), q_0 - 1)$$

Pairwise distinct permutations  $\Rightarrow$  at least  $q_1$   $h_1$ -words







$h_2$ -words

$$W_1^{(1)} = W_0^{(0)} \cdot W_{q_1-2}^{(0)} \cdot \dots \cdot W_1^{(0)} \cdot W_{q_1-1}^{(0)}$$

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$$\forall i \in \{0, \dots, q_2 - 1\}, W_i^{(1)} = W_0^{(0)} \cdot W_{\sigma_i^{(1)}(1)}^{(0)} \cdot W_{\sigma_i^{(1)}(2)}^{(0)} \cdot \dots \cdot W_{\sigma_i^{(1)}(q_1-2)}^{(0)} \cdot W_{q_1-1}^{(0)}$$

Pairwise distinct permutations and pairwise distinct subwords  $W_j^{(0)}$

$\Rightarrow$  at least  $q_2$   $h_2$ -words

## Proposition

If  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is non-decreasing, then  $\int_X \varphi(|c_T(x)|) d\mu(x)$  and  $\int_X \varphi(|c_S(x)|) d\mu(x)$  are bounded above by  $\sum_{n \geq 0} \frac{\varphi(q_0 \cdots q_{n+1})}{q_0 \cdots q_n}$ .

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## Corollary

With  $q_{n+1} = (q_n - 2)!$ , the cocycles are  $\log^\beta$ -integrable for every  $\beta < 1$ .

This concludes the proof of the theorem on entropy (the weaker version).

Topological version:

### Theorem (C. 2025+)

Let  $S$  be the universal odometer, let  $\alpha > 0$  or  $\alpha = +\infty$ . There exists a Cantor minimal homeomorphism  $T$  such that:

- $h_{\text{top}}(T) = \alpha$ ;
- $T$  and  $S$  are **strongly orbit equivalent**, with  $\log^\beta$ -integrable cocycles for every  $\beta < 1$ .

**Strong orbit equivalence:** the cocycles each have at most one point of discontinuity.

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Generalization of:

### Theorem (Boyle–Handelman 1994)

Let  $S$  the dyadic odometer, let  $\alpha > 0$  or  $\alpha = +\infty$ . There exists a Cantor minimal homeomorphism  $T$  such that:

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Sketch of proof of:

### Theorem (C. 2025+)

Let  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a sublinear map and let  $S$  be an odometer. Then there exists  $T \in \text{Aut}(X, \mu)$  such that  $T$  and  $S$  are  $\varphi$ -OE but not flip-conjugate.

We build  $T$  as an odomutant.

Reminder:  $\psi \circ T = S \circ \psi$  almost everywhere

$$\begin{aligned} \psi: X &\longrightarrow X \\ (x_n)_{n \geq 0} &\longmapsto \left( \sigma_{x_1}^{(0)}(x_0), \sigma_{x_2}^{(1)}(x_1), \sigma_{x_3}^{(2)}(x_2), \dots \right) = \lim_{n \rightarrow +\infty} \psi_n(x) \end{aligned}$$

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If  $\psi$  is invertible, then  $T$  and  $S$  are isomorphic (by definition)

### Question (Coalescence property)

If  $T$  and  $S$  are isomorphic, is  $\psi$  invertible?

**Yes!** because  $S$  is an odometer (**No!** in full generality)

It holds more generally when  $S$  is ergodic and has discrete spectrum.

## Definition

Let  $S \in \text{Aut}(X, \mu)$ . An *eigenfunction* of  $S$  is  $f \in L^2(X, \mu) \setminus \{0\}$  such that  $f \circ S = \lambda f$  for some  $\lambda \in \mathbb{C}$  (*eigenvalue*).

$E_\lambda(S) := \{\text{eigenfunctions associated to } \lambda\} \cup \{0\}$  (eigenspace).

If  $S$  is ergodic, then for every eigenvalue  $\lambda$ ,  $E_\lambda(S)$  is a line.

## Definition

$S \in \text{Aut}(X, \mu)$  has *discrete spectrum* if the span of all of its eigenfunctions is dense in  $L^2(X, \mu)$ . (example: odometers)

## Property

*If  $S$  is ergodic and has discrete spectrum, then  $S$  is coalescent: for every  $T \in \text{Aut}(X, \mu)$  isomorphic to  $S$ , every factor map  $\psi$  from  $T$  to  $S$  is invertible.*

Proof:  $f \in E_\lambda(S) \Rightarrow f \circ \psi \in E_\lambda(T)$

By ergodicity,  $E_\lambda(T) = E_\lambda(S) \circ \psi$

$T$  and  $S$  have discrete spectrum:  $L^2(X, \mu) = L^2(X, \mu) \circ \psi$ .

So  $\psi$  is invertible.

Sketch of proof of the theorem:

We build  $T$  as an odomutant such that

- $\psi$  is not invertible ( $T$  and  $S$  are not flip-conjugate);
- the distortions of the orbits are moderate ( $\varphi$ -integrable cocycles)