

Odomutants and flexibility results for quantitative orbit equivalence

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Analysis seminar – October 2, 2026

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- **Irrational rotation** of angle θ :

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- Finite alphabet Σ , prob. measure ν on Σ .

The **Bernoulli shift** on (Σ, ν) :

$$T: (x_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}},$$

$$\mu = \nu^{\otimes \mathbb{Z}}.$$

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$$S: \begin{array}{ccc} X & \longrightarrow & X \\ (x_n)_{n \geq 0} & \longmapsto & (x_n)_{n \geq 0} + (1, 0, 0, 0, \dots) \text{ with carry over to right} \end{array}$$

$$S(q_0 - 1, q_1 - 1, q_2 - 1, \dots) = (0, 0, 0, \dots).$$

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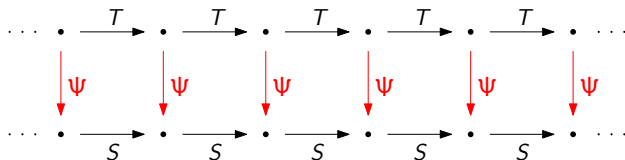
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But in full generality, the conjugacy problem is very hard!

Orbit equivalence

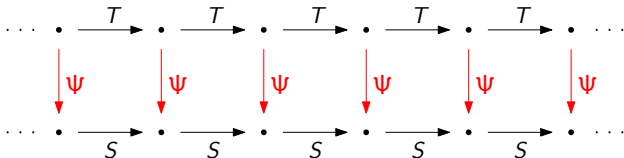
Conjugacy asks for $\Psi \in \text{Aut}(X, \mu)$ such that $\Psi(Tx) = S\Psi(x)$



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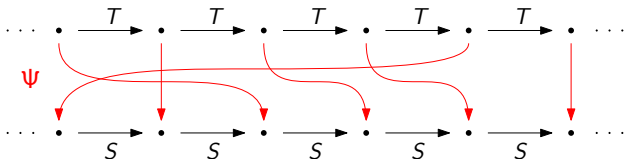
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Definition (Dye 1959)

T and $S \in \text{Aut}(X, \mu)$ are **orbit equivalent (OE)** if there exists $\Psi \in \text{Aut}(X, \mu)$ such that $\Psi(\text{Orb}_T(x)) = \text{Orb}_S(\Psi(x))$ for μ -a.e. $x \in X$.



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Converse:

Theorem (Dye 1959)

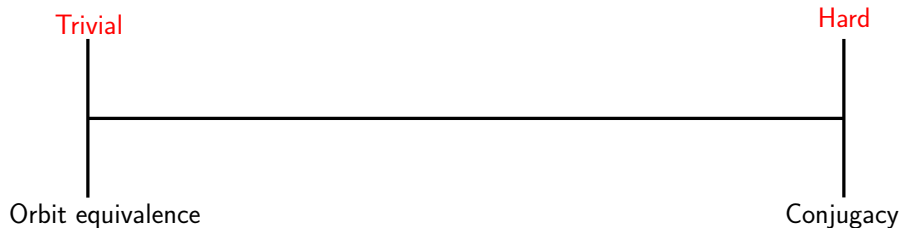
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Definition

$c_T: X \rightarrow \mathbb{Z}$ and $c_S: X \rightarrow \mathbb{Z}$ are the **cocycles** associated to the orbit equivalence Ψ .

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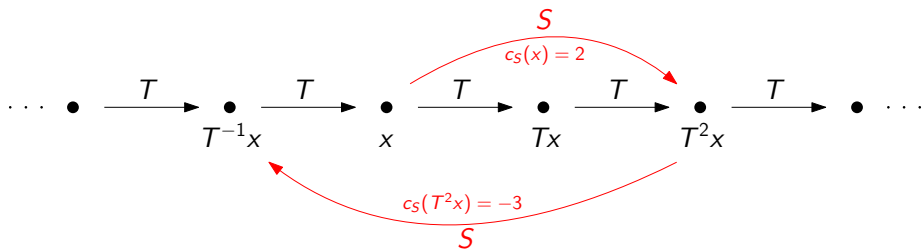
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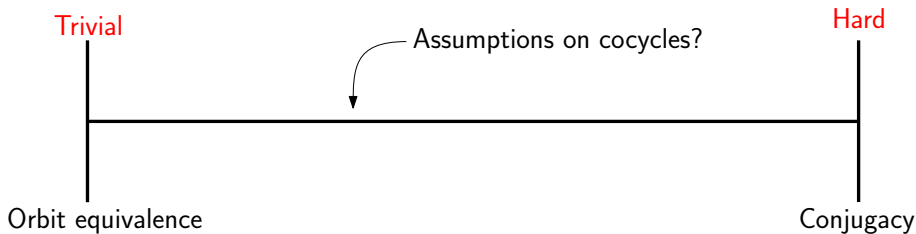
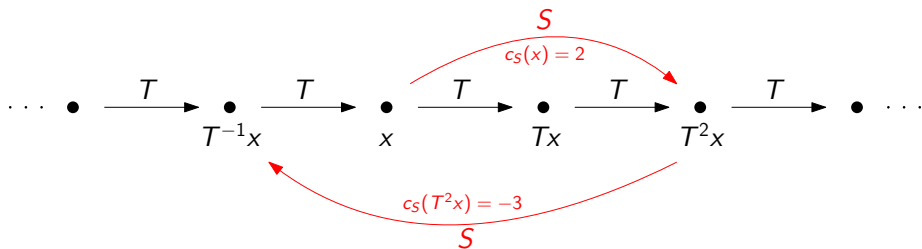
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Example with $\Psi = \text{id}_X$:

$$Tx = S^{c_T(x)}(x) \text{ and } Sx = T^{c_S(x)}(x)$$





Let $T, S \in \text{Aut}(X, \mu)$ be ergodic transformations, let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

Definition (Carderi–Joseph–Le Maître–Tessera 2023)

T and S are φ -**OE** if they admit an orbit equivalence whose cocycles are φ -integrable:

- $\int_X \varphi(|c_T(x)|) d\mu(x) < +\infty;$
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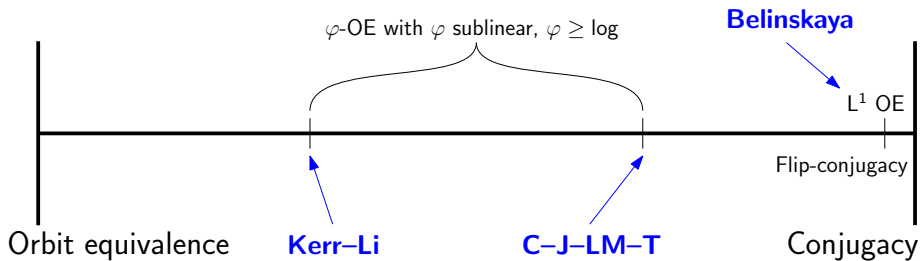
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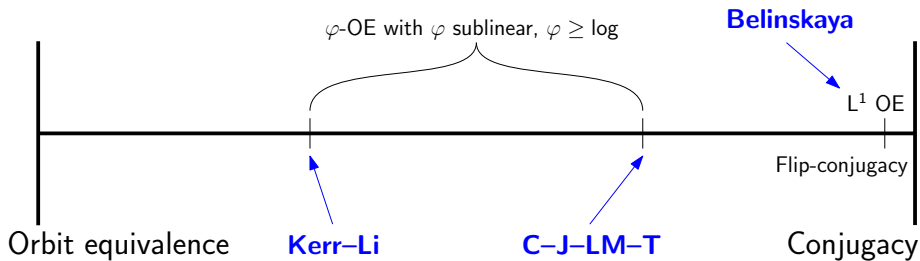
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L^p OE := φ -OE with $\varphi(x) = x^p$ ($p < 1$ is allowed!)

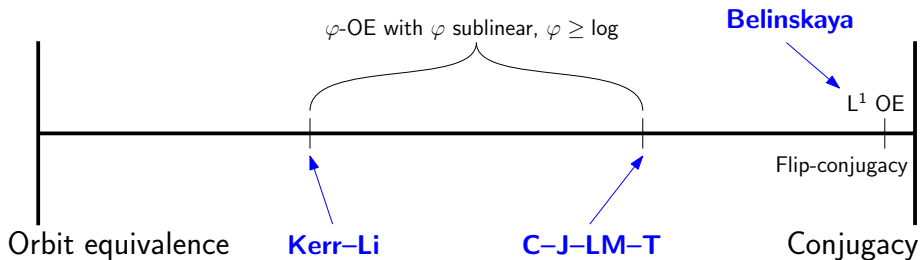
L^∞ OE $\Rightarrow$ exp-OE $\Rightarrow L^p$ OE $\Rightarrow$ log-OE





Theorem (Belinskaya 1969)

*If T and S are L^1 OE, then T is conjugate to S or S^{-1} (**flip-conjugacy**).*

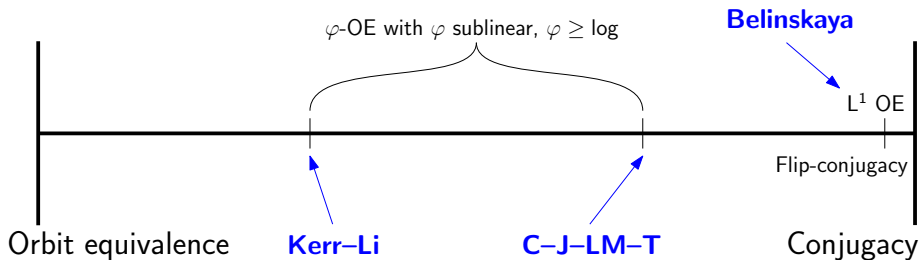


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Theorem (Kerr–Li 2023)

Entropy is preserved under log-OE.

Explicit constructions:

Theorem (Kerr–Li 2023, C. 2025)

Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\varphi(x) = o(x^{1/3})$.

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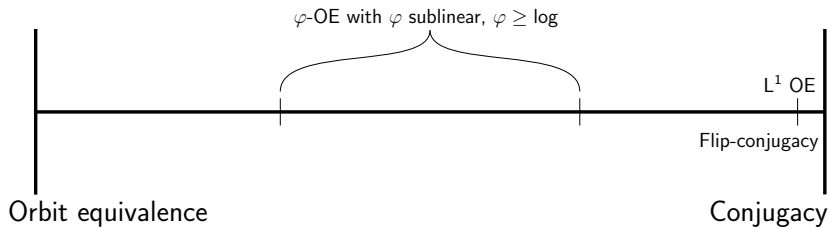
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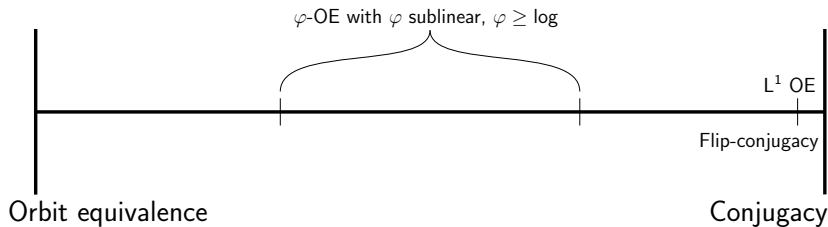
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Question

What about quantitative orbit equivalence between Bernoulli shifts?



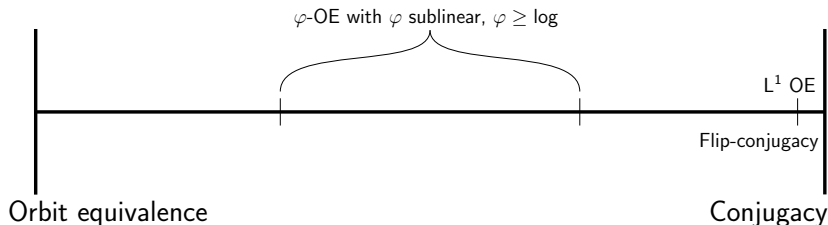


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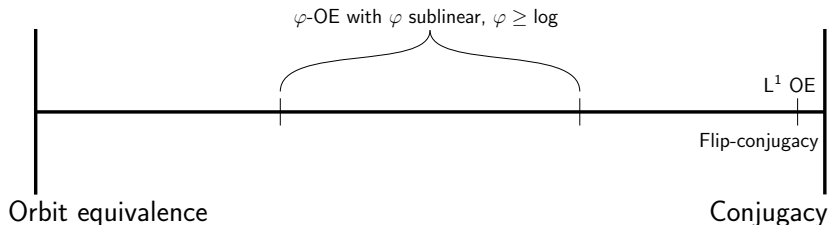
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Does there exist $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that any two ergodic transformations are φ -OE?

For instance: For $\beta < 1$, is $\log^\beta$ -OE trivial among ergodic transformations?
At least, we know that...

...Kerr and Li's theorem is optimal.

Theorem (C. 2026+)

Let S be an odometer with $k_{p_0} = \infty$ for some prime number p_0 . Let $\alpha > 0$ or $\alpha = +\infty$. There exists $T \in \text{Aut}(X, \mu)$ such that:

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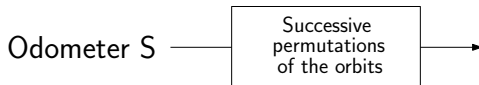
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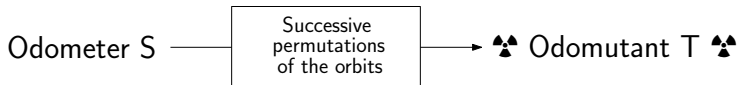


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Theorem (C. 2026+, weaker version)

Let S be the universal odometer. There exists $T \in \text{Aut}(X, \mu)$ such that:

- $h_\mu(T) > 0$;
- T and S are $\log^\beta$ -OE for all $\beta < 1$.

Another definition of odometer.

$$X = \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$$

Combinatorial structure:

Definition (n -cylinders)

Given $i_0 \in \{0, 1, \dots, q_0 - 1\}, \dots, i_{n-1} \in \{0, 1, \dots, q_{n-1} - 1\}$,

$$[i_0, \dots, i_{n-1}]_n := \{(x_i)_{n \geq 0} \in X \mid x_0 = i_0, \dots, x_{n-1} = i_{n-1}\}.$$

$$q_0 = 4$$

$[3]_1$	
$[2]_1$	
$[1]_1$	
$[0]_1$	

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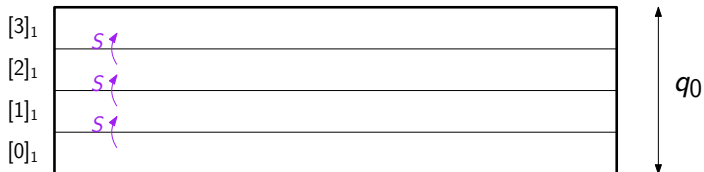
$[3]_1$	
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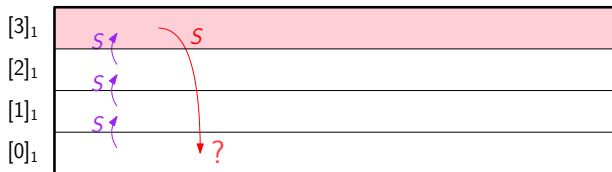
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s s

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 $\mathcal{R}_1$


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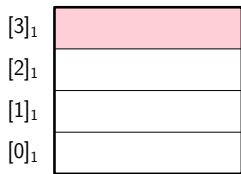
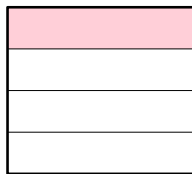
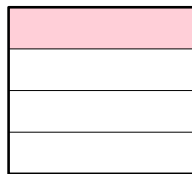
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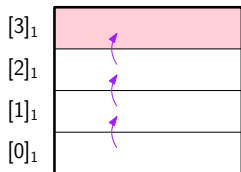
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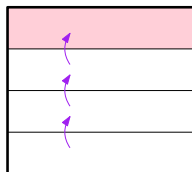
$$[\bullet, 2]_2$$

$$q_1 = 3$$

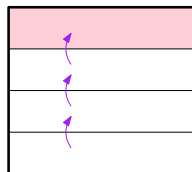

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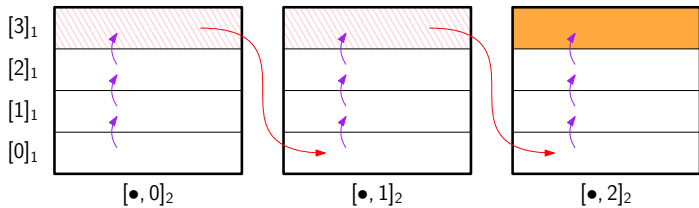
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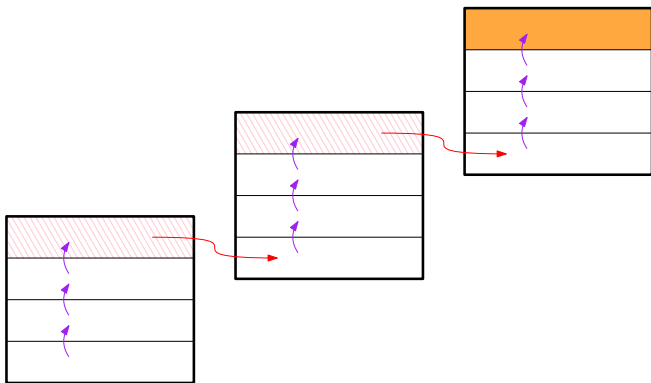


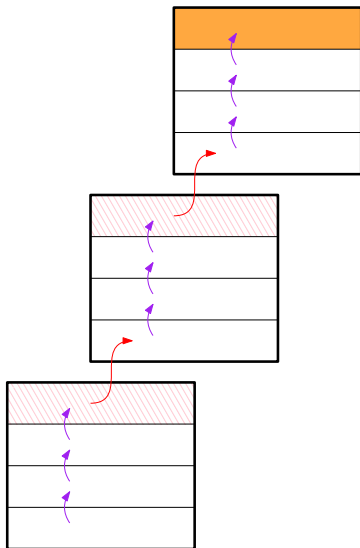
$[\bullet, 1]_2$

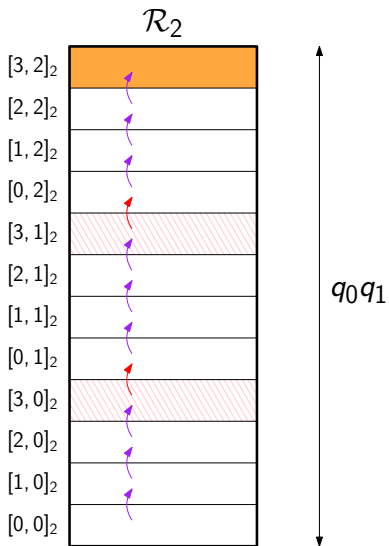


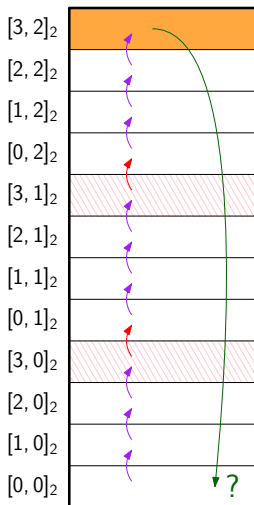
$[\bullet, 2]_2$







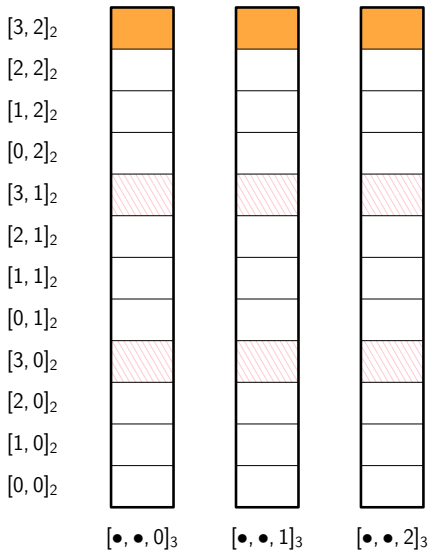


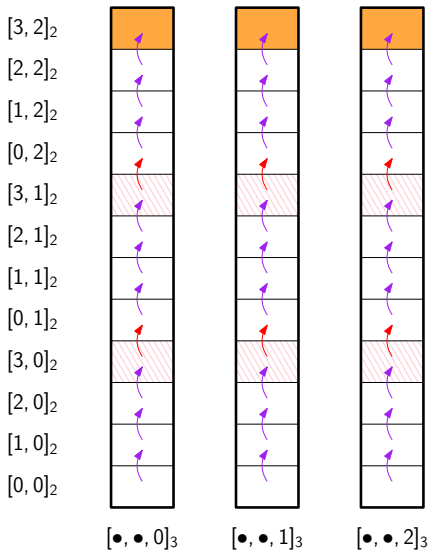


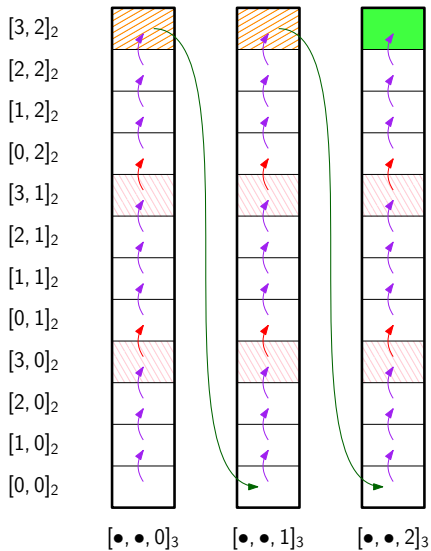
$[3, 2]_2$			
$[2, 2]_2$			
$[1, 2]_2$			
$[0, 2]_2$			
$[3, 1]_2$			
$[2, 1]_2$			
$[1, 1]_2$			
$[0, 1]_2$			
$[3, 0]_2$			
$[2, 0]_2$			
$[1, 0]_2$			
$[0, 0]_2$			

$[\bullet, \bullet, 0]_3$ $[\bullet, \bullet, 1]_3$ $[\bullet, \bullet, 2]_3$

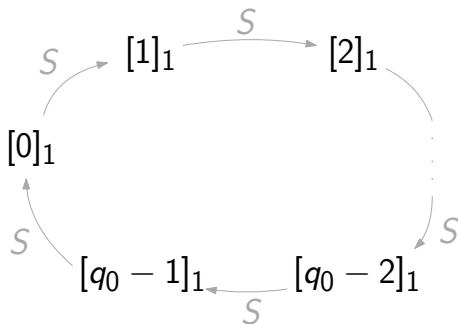
$q_2 = 3$







Odometer:



With an odomutant: less “predictable” dynamics

$$q_0 = 4$$

$[3]_1$	
$[2]_1$	
$[1]_1$	
$[0]_1$	

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$[3]_1$			
$[2]_1$			
$[1]_1$			
$[0]_1$			

$$[\bullet, 0]_2$$

$$[\bullet, 1]_2$$

$$[\bullet, 2]_2$$

$$q_1 = 3$$

$$q_0 = 4$$

$[3]_1$			
$[2]_1$			
$[1]_1$			
$[0]_1$			

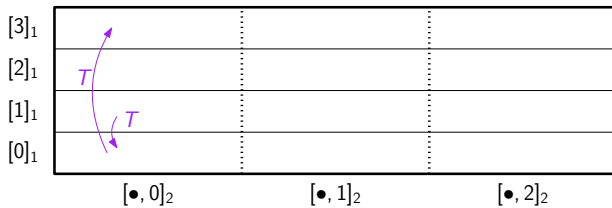
$$[\bullet, 0]_2$$

$$[\bullet, 1]_2$$

$$[\bullet, 2]_2$$

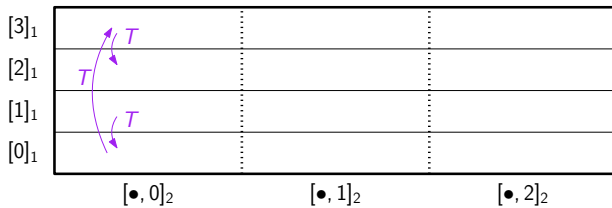
$$q_1 = 3$$

$$q_0 = 4$$



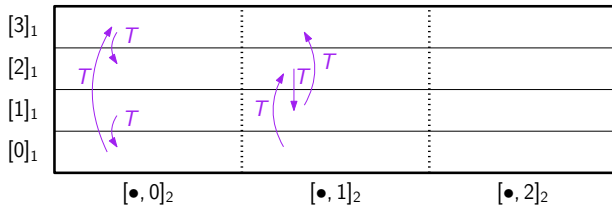
$$q_1 = 3$$

$$q_0 = 4$$



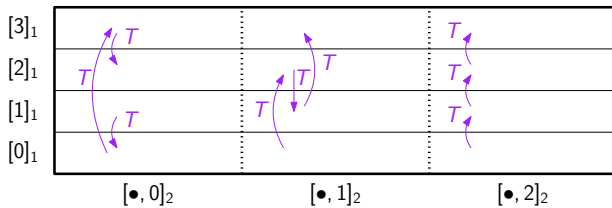
$$q_1 = 3$$

$$q_0 = 4$$



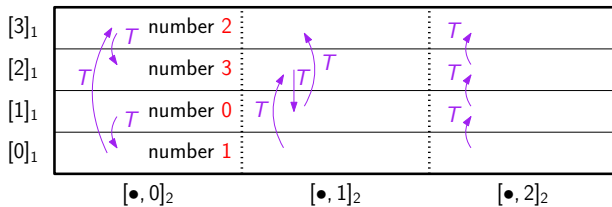
$$q_1 = 3$$

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$$q_1 = 3$$

$$q_0 = 4$$

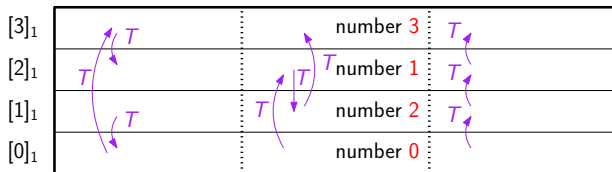


$$q_1 = 3$$

Permutations :

- 0 $\mapsto$ 1
- 1 $\mapsto$ 0
- 2 $\mapsto$ 3
- 3 $\mapsto$ 2

$$q_0 = 4$$



$$[\bullet, 0]_2$$

$$[\bullet, 1]_2$$

$$[\bullet, 2]_2$$

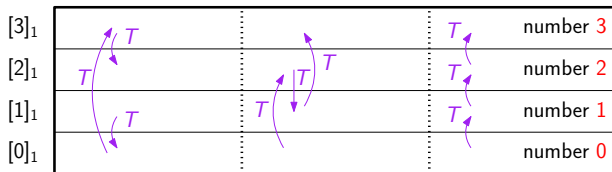
$$q_1 = 3$$

Permutations :

$$\begin{aligned} 0 &\mapsto 1 \\ 1 &\mapsto 0 \\ 2 &\mapsto 3 \\ 3 &\mapsto 2 \end{aligned}$$

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$$q_0 = 4$$



$$[\bullet, 0]_2$$

$$[\bullet, 1]_2$$

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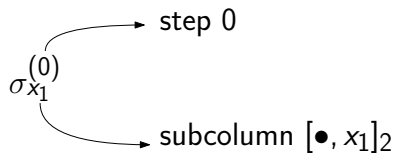
$$q_1 = 3$$

Permutations :

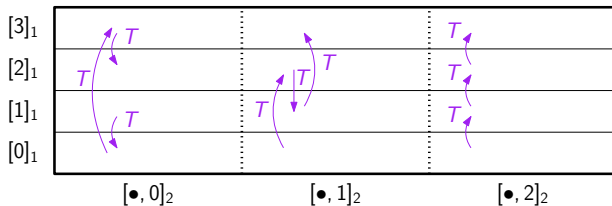
$$\begin{aligned} 0 &\mapsto 1 \\ 1 &\mapsto 0 \\ 2 &\mapsto 3 \\ 3 &\mapsto 2 \end{aligned}$$

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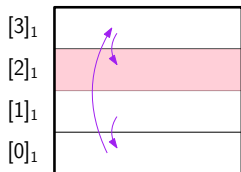


$q_1 = 3$

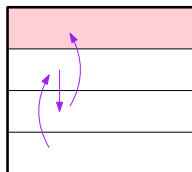
Permutations : $\sigma_0^{(0)}$:
 $0 \mapsto 1$
 $1 \mapsto 0$
 $2 \mapsto 3$
 $3 \mapsto 2$

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 $0 \mapsto 0$
 $1 \mapsto 2$
 $2 \mapsto 1$
 $3 \mapsto 3$

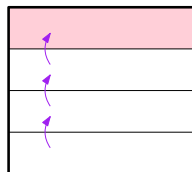
$\sigma_2^{(0)}$:
 $0 \mapsto 0$
 $1 \mapsto 1$
 $2 \mapsto 2$
 $3 \mapsto 3$



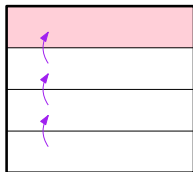
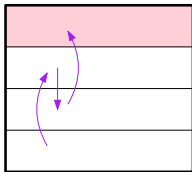
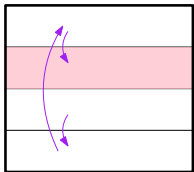
$[\bullet, 0]_2$

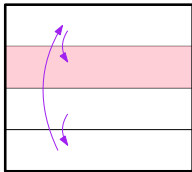
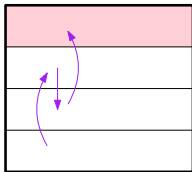
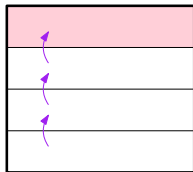


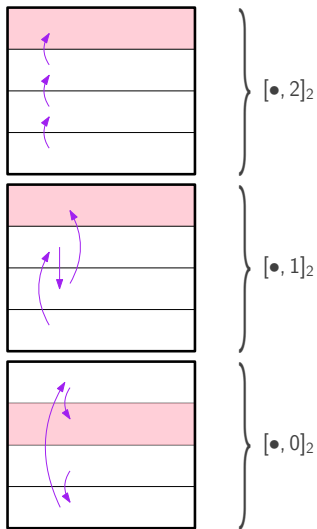
$[\bullet, 1]_2$

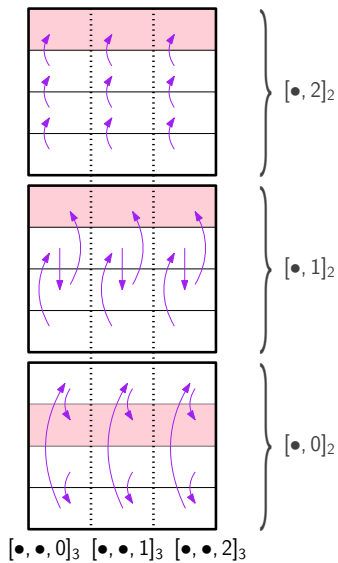


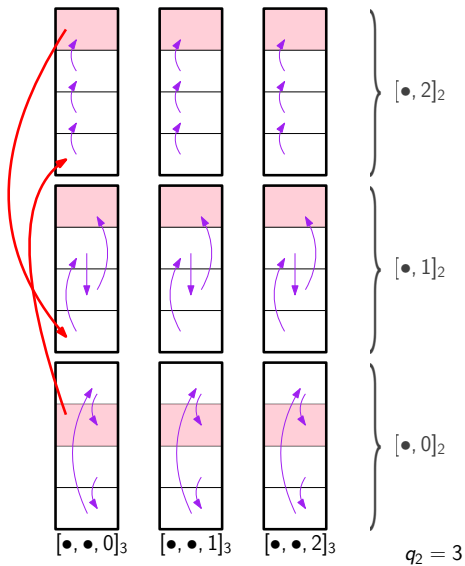
$[\bullet, 2]_2$







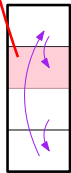
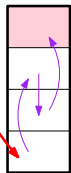
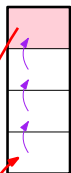




number 1

number 2

number 0



$[\bullet, \bullet, 0]_3$

$0 \mapsto 0$

$1 \mapsto 2$

$2 \mapsto 1$



$[\bullet, \bullet, 1]_3$



$[\bullet, \bullet, 2]_3$

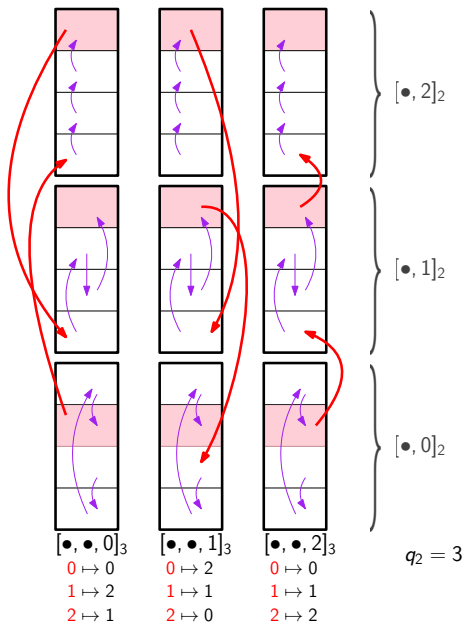
$[\bullet, 2]_2$

$[\bullet, 1]_2$

$[\bullet, 0]_2$

$q_2 = 3$

Permutations :

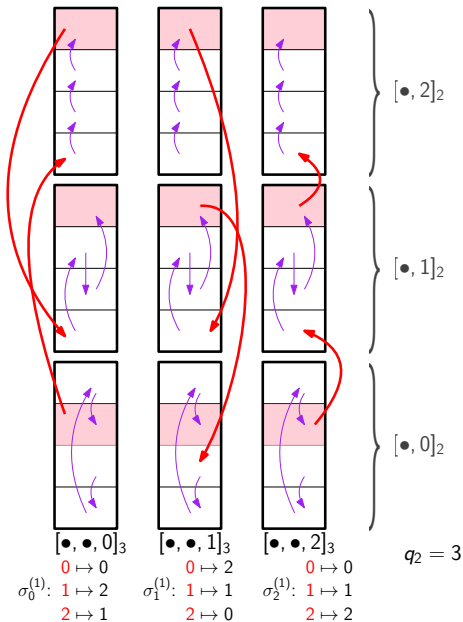


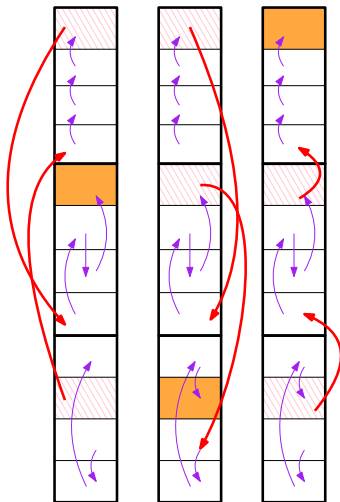
Permutations :

$\sigma_{x_2}^{(1)}$

 step 1

 subcolumn $[\bullet, \bullet, x_2]_2$





An odomutant T is the data of

- **integers** $q_0, q_1, q_2 \dots \geq 2$
and the odometer S on $X = \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$;

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At step n :

x_{n+1} $(n+1)$ -th coordinate $\in \{0, \dots, q_{n+1} - 1\}$
represents the block $[\bullet, \dots, \bullet, \bullet, x_{n+1}]_{n+2}$

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→ permutation of sub-blocks

$$\begin{aligned}
 & [\bullet, \dots, \bullet, 0, x_{n+1}]_{n+2}, \\
 & [\bullet, \dots, \bullet, 1, x_{n+1}]_{n+2}, \\
 & \vdots \\
 & [\bullet, \dots, \bullet, q_n - 1, x_{n+1}]_{n+2}
 \end{aligned}$$

$$\begin{aligned} \psi_n: \quad X &\longrightarrow X \\ (x_n)_{n \geq 0} &\longmapsto \left(\sigma_{x_1}^{(0)}(x_0), \sigma_{x_2}^{(1)}(x_1), \dots, \sigma_{x_{n+1}}^{(n)}(x_n), x_{n+1}, x_{n+2}, \dots \right) \end{aligned}$$

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$$T: \{x \in X \mid \psi(x) \neq (q_0 - 1, q_1 - 1, \dots)\} \longrightarrow \{x \in X \mid \psi(x) \neq (0, 0, \dots)\}$$

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- $T \in \text{Aut}(X, \mu)$;
- T and S have the same orbits (up to a null set);
- $\psi \circ T = S \circ \psi$ almost everywhere,
so T factorizes onto S .

Proof of:

Theorem (C. 2026+, weaker version)

Let S be the universal odometer. There exists $T \in \text{Aut}(X, \mu)$ such that:

- $h_\mu(T) > 0$;
- T and S are $\log^\beta$ -OE for all $\beta < 1$.

Calculation of $h_\mu(T)$:

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Calculation of $h_\mu(T)$:

- easier to compute topological entropy
→ $T: X \rightarrow X$ has to be continuous X
- apply the variational principle
→ If T is uniquely ergodic, then $h_{\text{top}}(T) = h_\mu(T)$

HYPOTHESIS 1:

For every $n \geq 0$,

0 et $q_n - 1$ are fixed point of the permutations $\sigma_0^{(n)}, \sigma_1^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$.

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Property

- $T: X \rightarrow X$ is a homeomorphism (so is continuous).
- For every $x \in X$ (not only "up to a null set"),
 $\text{Orb}_T(x) = \text{Orb}_S(x)$ (so T is uniquely ergodic).

Goal:

- find the parameters:
 - the integers $\mathbf{q}_n$ (and the underlying odometer S);
 - and the permutations $\sigma_i^{(n)}$ (satisfying **HYPOTHESIS 1**),such that $h_{\text{top}}(T) > 0$;

Goal:

- find the parameters:
 - the integers q_n (and the underlying odometer S);
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such that $h_{\text{top}}(T) > 0$;
- ...in the hope of getting an orbit equivalence which is $\log^\beta$ -integrable for every $\beta < 1$.

$$\ln X = \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\} \dots$$

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Reminder

For $i \in \{0, 1, \dots, q_0 - 1\}$, $[i]_1 = \{(x_n)_{n \geq 0} \in X \mid x_0 = i\}$ (1-cylinder)

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Remark

Given $x \in X$,

$$\begin{aligned} x &\in [i_0]_1 \cap T^{-1}([i_1]_1) \cap \dots \cap T^{-(n-1)}([i_{n-1}]_1) \\ \iff x &\in [i_0]_1, \quad Tx \in [i_1]_1, \quad \dots, \quad T^{n-1}x \in [i_{n-1}]_1 \end{aligned}$$

$\leadsto$ To x , we associate the n -word $(i_0, \dots, i_{n-1})$

$$h_{\text{top}}(T) \geq \lim_{n \rightarrow +\infty} \frac{\log(\text{number of } n\text{-words } (i_0, \dots, i_{n-1}) \text{ obtained from 1-cylinders})}{n}$$

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HYPOTHESIS 2:

For every $n \geq 0$,

$\sigma_0^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$ describe all the permutations of $\{0, 1, \dots, q_n - 1\}$ fixing 0 and $q_n - 1$, and are pairwise distinct.

$$\leadsto q_{n+1} = (q_n - 2)!$$

Reminder on **HYPOTHESIS 1**:

For every $n \geq 0$,

0 and $q_n - 1$ are fixed points of the permutations $\sigma_0^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$.

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Proposition

With **HYPOTHESIS 2**, we produce a lot of words!

$$h_{\text{top}}(T) \geq (\log q_0) - C \text{ where } C \text{ is a constant.}$$

If q_0 is large enough, then $h_{\text{top}}(T) > 0$.

Proof of the proposition:

① We prove that

"number of h_n -words obtained from 1-cylinders" $\geq q_n$

where $h_{n+1} = q_0 q_1 \dots q_n$ is the height of the towers at step n .

(Proof by drawings in the next two slides, cases $n = 0$ and $n = 1$)

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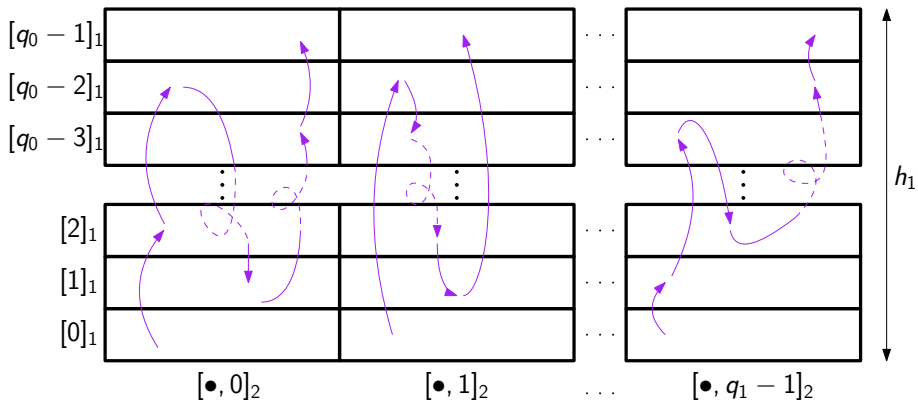
(Proof by drawings in the next two slides, cases $n = 0$ and $n = 1$)

②
$$h_{\text{top}}(T) \geq \lim_{n \rightarrow +\infty} \frac{\log(\text{number of } n\text{-words obtained from 1-cylinders})}{n},$$

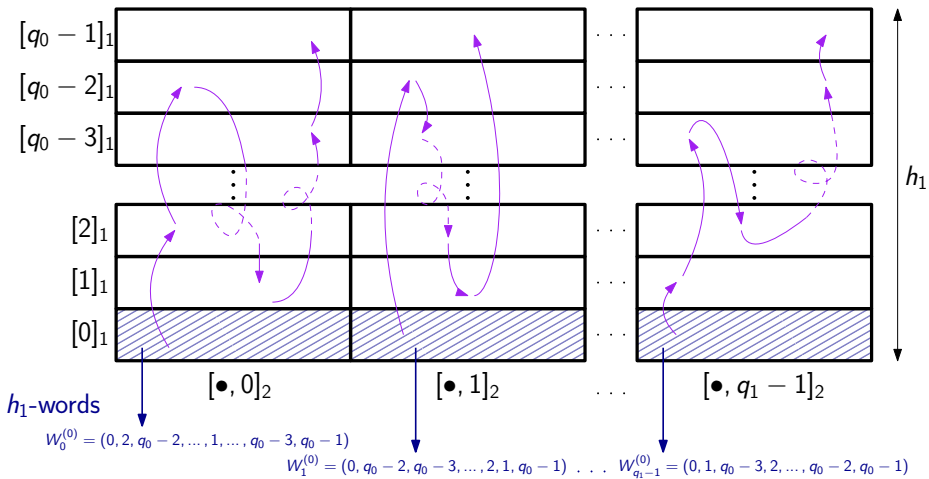
so
$$h_{\text{top}}(T) \geq \lim_{n \rightarrow +\infty} \frac{\log q_n}{h_n}.$$

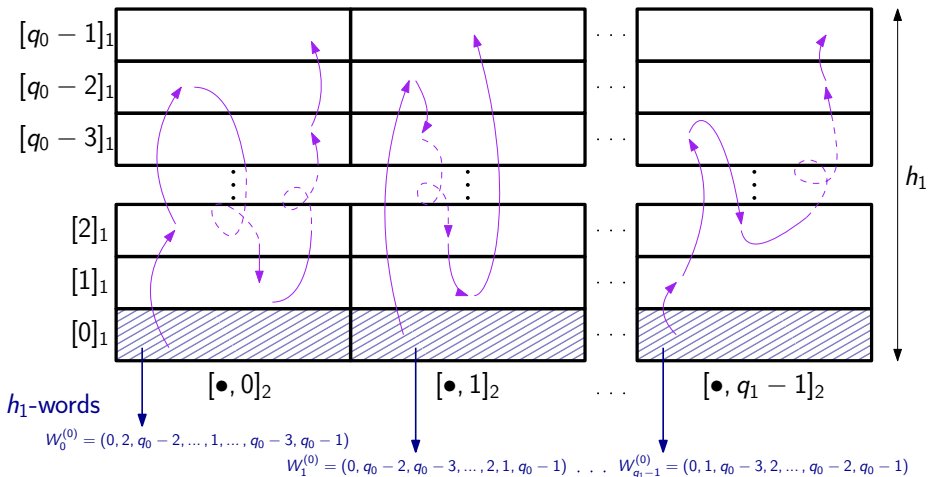
To get $\frac{\log q_n}{h_n} \geq (\log q_0) - C$, we successively use

- $q_{i+1} = (q_i - 2)!;$
- $\log(q_i!) \geq q_i \log q_i;$
- $h_{i+1} = q_i h_i.$



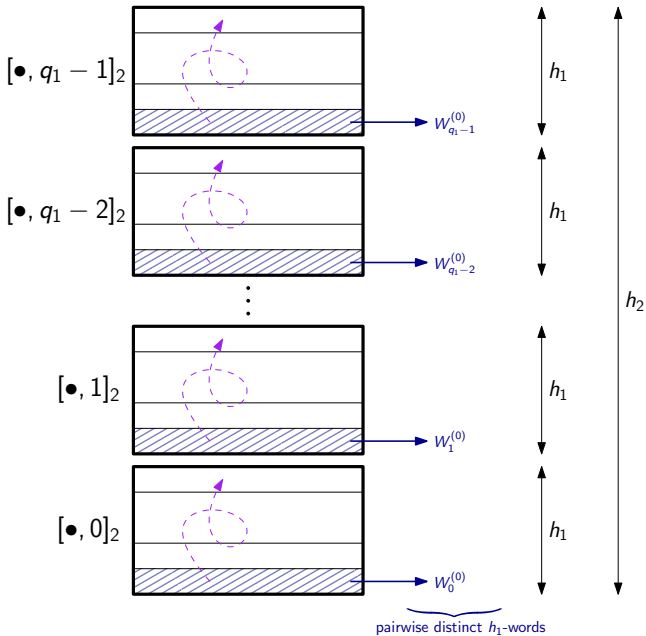
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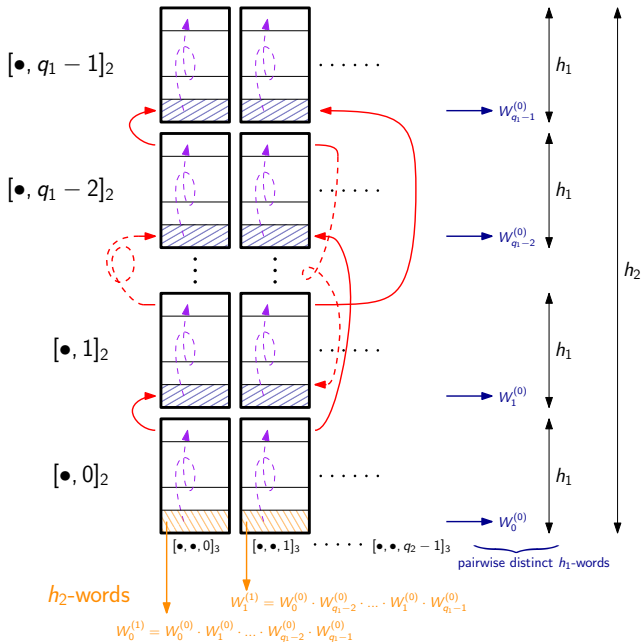


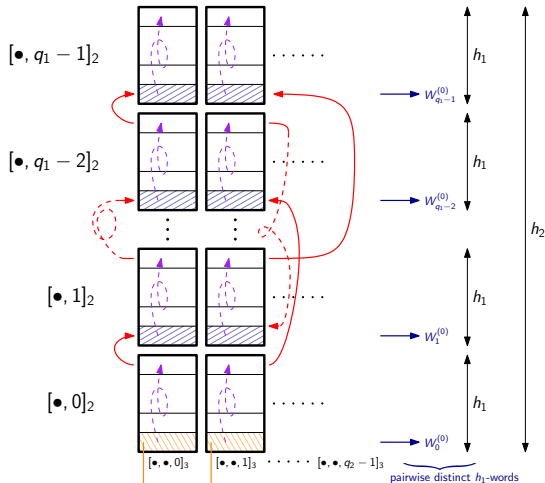


$$\forall i \in \{0, \dots, q_1 - 1\}, W_i^{(0)} = (0, \sigma_i^{(0)}(1), \sigma_i^{(0)}(2), \dots, \sigma_i^{(0)}(q_0 - 2), q_0 - 1)$$

Pairwise distinct permutations $\Rightarrow$ at least q_1 h_1 -words







h_2 -words

$$W_1^{(1)} = W_0^{(0)} \cdot W_{q_1-2}^{(0)} \cdot \dots \cdot W_1^{(0)} \cdot W_{q_1-1}^{(0)}$$

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$$\forall i \in \{0, \dots, q_2 - 1\}, W_i^{(1)} = W_0^{(0)} \cdot W_{\sigma_i^{(1)}(1)}^{(0)} \cdot W_{\sigma_i^{(1)}(2)}^{(0)} \cdot \dots \cdot W_{\sigma_i^{(1)}(q_1-2)}^{(0)} \cdot W_{q_1-1}^{(0)}$$

Pairwise distinct permutations and pairwise distinct subwords $W_j^{(0)}$
 $\Rightarrow$ at least q_2 h_2 -words

Proposition

If $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is non-decreasing, then $\int_X \varphi(|c_T(x)|) d\mu(x)$ and $\int_X \varphi(|c_S(x)|) d\mu(x)$ are bounded above by $\sum_{n \geq 0} \frac{\varphi(q_0 \dots q_{n+1})}{q_0 \dots q_n}$.

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Corollary

With $q_{n+1} = (q_n - 2)!$, the cocycles are $\log^\beta$ -integrable for every $\beta < 1$.

This concludes the proof of the theorem on entropy (the weaker version).

Topological version:

Theorem (C. 2026+)

Let S be an odometer with $k_{p_0} = \infty$ for some prime number p_0 . Let $\alpha > 0$ or $\alpha = +\infty$. There exists a Cantor minimal homeomorphism T such that:

- $h_{\text{top}}(T) = \alpha$;
- T and S are **strongly orbit equivalent**, with $\log^\beta$ -integrable cocycles for every $\beta < 1$.

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Generalization of:

Theorem (Boyle–Handelman 1994)

Let S be the dyadic odometer, let $\alpha > 0$ or $\alpha = +\infty$. There exists a Cantor minimal homeomorphism T such that:

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Same construction, with different ways to establish the orbit equivalence:

- in Boyle and Handelman's proof: implicit way (complete invariant: dimension group);
- with odomutants: explicit construction.

Other topological interest:

Hypothesis 1 $\Rightarrow$ odomutant T is an almost 1-to-1 extension of the odometer S .

Hypothesis 1 & “pairwise different” in **Hypothesis 2** $\Rightarrow T$ is a subshift

$\rightarrow$ We have built Toeplitz systems.