XV Ukraine Algebra Conference, July 8-12, 2025

Isoperimetric profile and quantitative orbit equivalence for lamplighter-like groups

joint work with Vincent Dumoncel

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10-07-2025

Groups as metric spaces

If S_G generates the group G, let us define for every $g \in G$:

$$|g|_G \coloneqq \min \{n \geq 0 \mid \exists s_1, \ldots, s_n \in S_G \cup S_G^{-1}, g = s_1 \ldots s_n\}.$$

Metric: $d_{S_G}(g, g') = |gg'^{-1}|_{S_G}$

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Fact

If S_G and S'_G are two finite generating sets of G, then there exists $C > 0$ such that for every $g \in G$,

$$\frac{1}{C}|g|_{S'_G} \leq |g|_{S_G} \leq C|g|_{S'_G}.$$

Goal: study of the large-scale geometry of **finitely generated** groups.

Groups as metric spaces

How to geometrically compare finitely generated groups?
$\rightsquigarrow$ **quasi-isometry**

Example

$\mathbb{Z}^k$ and $\mathbb{Z}^d$ are quasi-isometric if and only if $k = d$.

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Question

When G and H are not quasi-isometric, how much do their geometries differ?

$\rightsquigarrow$ **Quantitative orbit equivalence** offers a more quantitative comparison between amenable groups

We focus on finitely generated amenable groups.

Orbit equivalence and associated cocycles

(X, μ): standard and atomless probability space ($\cong ([0,1], \mathrm{Leb})$)
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Two groups G and H are **orbit equivalent** (OE)
if there exists free p.m.p. G- and H-actions on (X,μ)
having the same orbits: for almost every $x \in X$, $\boldsymbol{G \cdot x = H \cdot x}$.

(X,μ) is called an **orbit equivalence coupling** between G and H.

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Definition

$c_{G,H} \colon G \times X \to H$ and $c_{H,G} \colon H \times X \to G$ are the **cocycles** of this coupling.

Quantitative orbit equivalence

Theorem (Ornstein, Weiss 1980)

Any two infinite amenable groups are OE.

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Example

- $(\mathrm{L}^p, \mathrm{L}^q)$ orbit equivalence:
  $|c_{G,H}(g,.)|_H \colon X \to \mathbb{R}_+$ is L^p, $|c_{H,G}(h,.)|_G \colon X \to \mathbb{R}_+$ is L^q

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  $|c_{G,H}(g,.)|_H \colon X \to \mathbb{R}_+$ is L^p, $|c_{H,G}(h,.)|_G \colon X \to \mathbb{R}_+$ is L^q
- $(\log, \mathrm{L}^0)$-integrable orbit equivalence:
  $\log(|c_{G,H}(g,.)|_H) \colon X \to \mathbb{R}_+$ is integrable, no requirement on $c_{H,G}$
- $(\exp, \log^{\circ n})$-integrable orbit equivalence, etc.

A measurement of amenability: isoperimetric profile

Definition

The **isoperimetric profile** $j_{1,G}\colon \mathbb{N} \to \mathbb{R}_+$ of a finitely generated group G is:

$$j_{1,G}(n) \coloneqq \sup_{\substack{A \subset G \\ |A| \leq n}} \frac{|A|}{|\partial_G A|}$$

where $\partial_G A \coloneqq (S_G A) \setminus A$ is the boundary of A.

G is amenable if and only if $j_{1,G}$ is unbounded

"The faster the isoperimetric profile tends to infinity, the more the group is amenable"

How integrable cocycles can be?

Theorem (Delabie, Koivisto, Le Maître, Tessera 2023)

Let G and H be finitely generated groups.
Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing map such that $t \mapsto t/\varphi(t)$ is increasing.

If there exists a (φ, L^0)-integrable OE coupling from G to H, then

$$\varphi(j_{1,H}(x)) \preccurlyeq j_{1,G}(x).$$

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Application:

Theorem (Delabie, Koivisto, Le Maître, Tessera 2023, C. 2025)

There exists an $(\mathrm{L}^p, \mathrm{L}^0)$-integrable OE coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ if and only if $p < \frac{k}{k+\ell}$.

Quantitative orbit equivalence finely compares the geometry of the groups $\mathbb{Z}^d$.

Our joint work

Let's talk about our joint work with Vincent Dumoncel

- we consider **lampshuffler groups**

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- we consider **lampshuffler groups**
- we build **orbit equivalence** couplings between these groups
- we **quantify** the cocycles
- we compute the **isoperimetric profiles** to prove quantitative **optimality** of these couplings.

Lampshuffler groups

Definition

Given a group H,
$\mathrm{FSym}(H) :=$ set of permutations $\sigma\colon H \to H$ of finite support
(i.e. $\{h \in H \mid \sigma(h) \neq h\}$ is finite)
$\mathsf{Shuffler}(H) := \mathrm{FSym}(H) \rtimes H$

Notation: $\mathsf{Shuffler}^{\circ 0}(H) = H$, $\mathsf{Shuffler}^{\circ(n+1)}(H) = \mathsf{Shuffler}^{\circ n}(\mathsf{Shuffler}(H))$

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Genevois and Tessera initiated a classification up to quasi-isometry of lampshufflers.

Question

When two lampshufflers are not quasi-isometric, how much their geometries differ?

For simplicity, we explain our results for the groups $\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^d)$.

Isoperimetric profiles of lampshufflers

Theorem (Erschler, Zheng 2023)

$$j_{1,\mathsf{Shuffler}(\mathbb{Z}^d)}(x) \simeq \left(\frac{\log(x)}{\log(\log(x))}\right)^{1/d}$$

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For general finitely generated amenable groups H, bounds for $j_{1,\mathsf{Shuffler}(H)}$ have been found [Saloff-Coste - Zheng 2021, Erschler - Zheng 2023], **not sharp** in full generality.
For instance, $j_{1,\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^d)}$ was not known.

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Theorem (C., Dumoncel 2025+)

$$j_{1,\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^d)}(x) \simeq \left(\frac{\log^{\circ n}(x)}{\log^{\circ(n+1)}(x)}\right)^{1/d}$$

We get the isoperimetric profiles of $\mathsf{Shuffler}^{\circ n}(H)$ for more general finitely generated amenable groups H (with mild assumptions).

Quasi-isometry and quantitative orbit equivalence

Theorem (C., Dumoncel 2025+)

$\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^d)$ *and* $\mathsf{Shuffler}^{\circ m}(\mathbb{Z}^k)$ *are quasi-isometric if and only if* $n = m$ *and* $d = k$.

Quasi-isometry and quantitative orbit equivalence

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Quantitative comparison:

Theorem (C., Dumoncel 2025+)

If $m > n$, *there exists a* $((\log^{\circ \ell})^p, \mathrm{L}^0)$*-integrable OE coupling from* $\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^k)$ *to* $\mathsf{Shuffler}^{\circ m}(\mathbb{Z}^d)$ *if and only if* $p < \frac{1}{k}$ *and* $\ell \leq m - n$.

We get more general results on the quantitative orbit equivalence between lampshufflers.

Thank you for listening !

[CD] Corentin Correia and Vincent Dumoncel. "On quantitative orbit equivalence between Lamplighter-like Groups". In preparation.

[CD25] Corentin Correia and Vincent Dumoncel. Isoperimetric Profiles of Lamplighter-like Groups. 2025.

[DKLT22] T. Delabie, J. Koivisto, F. Le Maître, and R. Tessera. "Quantitative Measure Equivalence between Amenable Groups". In: Annales Henri Lebesgue 5 (2022), pp. 1417–1487.

[GT24] Anthony Genevois and Romain Tessera. Lamplighter-like Geometry of Groups. 2024.