Odomutants and quantitative orbit equivalence

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(X, μ) : standard and atomless probability space

$\cong ([0,1], \mathrm{Leb})$

$\mathrm{Aut}(X, \mu) := \{\text{bimeasurable bijections } T \colon X \to X,\ \mu(T^{-1}(.)) = \mu(.)\}$

$\rightsquigarrow \mathbb{Z}$-actions

Definition

Two transformations $S, T \in \mathrm{Aut}(X, \mu)$ are **orbit equivalent**
if there exists $\theta \in \mathrm{Aut}(X, \mu)$
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Let $S, T \in \mathrm{Aut}(X,\mu)$ be aperiodic transformations.

If S et $\theta^{-1}T\theta$ have the same orbits :

↙ ↘

$Sx \in \left\{(\theta^{-1}T\theta)^n(x) \mid n \in \mathbb{Z}\right\}$
so $Sx = (\theta^{-1}T\theta)^{c_S(x)}(x)$

$Tx \in \left\{(\theta S\theta^{-1})^n(x) \mid n \in \mathbb{Z}\right\}$
so $Tx = (\theta S\theta^{-1})^{c_T(x)}(x)$

$c_S\colon X \to \mathbb{Z}$ and $c_T\colon Y \to \mathbb{Z}$ are called the **cocycles** associated to θ.

Definition

Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$. S and T are **φ-integrably orbit equivalent** if there exists an orbit equivalence $\theta \in \mathrm{Aut}(X, \mu)$ and **the associated cocycles are φ-integrable** :

$$\int_X \varphi(|c_S(x)|)\mathrm{d}\mu(x) < \infty \text{ and } \int_X \varphi(|c_T(x)|)\mathrm{d}\mu(x) < \infty$$

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Theorem (David Kerr, Hanfeng Li - 2023)

If S, T are φ-integrably orbit equivalent with $\boldsymbol{\varphi \geq \log}$, *then* $\mathbf{h}_{\boldsymbol{\mu}}\boldsymbol{(S) =} \mathbf{h}_{\boldsymbol{\mu}}\boldsymbol{(T)}$.

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Theorem (C. - 2024)

There exist S and T satisfying :

1. $\mathrm{h}_\mu(S) = 0$; $\mathrm{h}_\mu(T) > 0$;
2. *S and T are $\boldsymbol{\log^\alpha}$-integrably orbit equivalent for all $\boldsymbol{\alpha < 1}$.*

Odometer S

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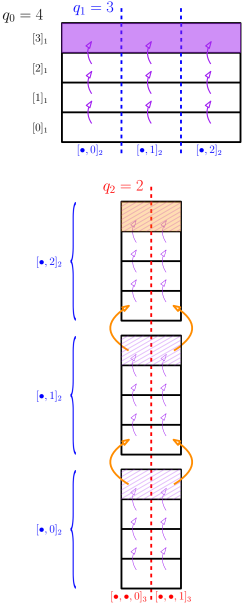
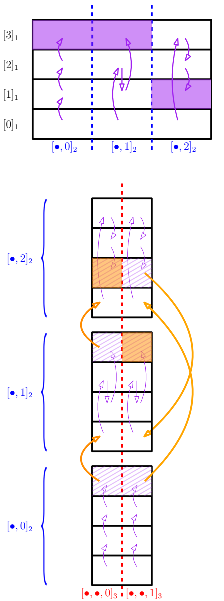
Odometer S $\rightsquigarrow$ ☢ Odomutant T ☢

The odometer S on $\prod_{n\geq 0}\{0,\dots,q_n-1\}$



An odomutant T associated to S



(No odometers were harmed during this work)