Isoperimetric profile and quantitative orbit equivalence for lamplighter-like groups

joint work with Vincent Dumoncel

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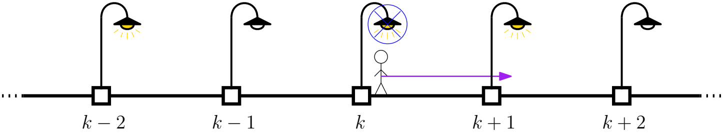

(Illustration of the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$)

Framework

(X, μ): standard and atomless probability space
$\cong ([0,1], \mathrm{Leb})$

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Let G be a group.

- A **p.m.p.** G-action on (X,μ):
  action of G on (X,μ) by bimeasurable bijections preserving the probability measure μ.
- The action is (essentially) **free**:
  for almost every $x \in X$, $\forall g \in G$, $[g \cdot x = x \Longrightarrow g = 1_G]$.

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Example

Bernoulli shift $G \curvearrowright [0,1]^G$

$$g \cdot (\varepsilon_h)_{h \in G} = (\varepsilon_{g^{-1}h})_{h \in G}$$

It is a free p.m.p. G-action on $([0,1]^G, \mathrm{Leb}^{\otimes G})$.

Amenability

Definition

A countable group G is **amenable** if
there exists a sequence $(F_n)_{n\geq 0}$ of finite subsets of G (**Følner** sequence)
such that $\forall g \in G$, $\frac{|gF_n \setminus F_n|}{|F_n|} \underset{n\to+\infty}{\longrightarrow} 0$.

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Example

- Finite groups F ($F_n = F$)
- $\mathbb{Z}^d$ ($F_n = \{0, \ldots, n\}^d$)
- Solvable groups are amenable
- Free groups are not amenable

Orbit equivalence

Definition

Two groups G and H are **orbit equivalent** (OE)
if there exists free p.m.p. G- and H-actions on (X, μ)
having the same orbits: for almost every $x \in X$, $\boldsymbol{G \cdot x = H \cdot x}$.

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Flexibility in the **amenable** world!

Theorem (Ornstein, Weiss 1980)

Any two infinite amenable groups are OE.

Cocycles

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Definition

$c_{G,H} \colon G \times X \to H$ and $c_{H,G} \colon H \times X \to G$ are the **cocycles** of this coupling.

Cocycles

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$x \in X \mapsto c_{G,H}(.,x)$ is a random bijection from G to H

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Assume $G = \langle S_G \rangle$ and $H = \langle S_H \rangle$. Let $g \in S_G$.
Is $c_{G,H}(g,x)$ in S_H ? not far from being in S_H ?

If $H = \langle S_H \rangle$, let us define for every $h \in H$:

$$|h|_H \coloneqq \min\{n \geq 0 \mid \exists s_1, \dots, s_n \in S_H \cup S_H^{-1}, h = s_1 \dots s_n\}.$$

Example

For $H = \mathbb{Z}$ with $S_H = \{1\}$, we have $|.|_{\mathbb{Z}}$ = absolute value

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Example

For $H = \mathbb{Z}$ with $S_H = \{1\}$, we have $|.|_{\mathbb{Z}} =$ absolute value

$|h|_H = 0 \iff h = 1_H$
$|h|_H = 1 \iff h \in (S_H \cup S_H^{-1}) \setminus \{1_H\}$

Goal: study of $|c_{G,H}(g,.)|_H$ and $|c_{H,G}(h,.)|_G$ for every $g \in S_G$ and $h \in S_H$.

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- $\forall g \in S_G,\ \exists C_g > 0,\ \int_X \varphi(C_g |c_{G,H}(g,x)|_H) \mathrm{d}\mu(x) < +\infty;$

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Examples: $(\log, \mathrm{L}^0)$, $(\mathrm{L}^0, \mathrm{L}^{1/2})$, ...

Isoperimetric profile

Definition

The **isoperimetric profile** $j_{1,G}\colon \mathbb{N} \to \mathbb{R}_+$ of a finitely generated group G is:

$$j_{1,G}(n) \coloneqq \sup_{\substack{A \subset G \\ |A| \leq n}} \frac{|A|}{|\partial_G A|}$$

where $\partial_G A \coloneqq (S_G A)/A$ is the boundary of A.

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G is amenable if and only if $j_{1,G}$ is unbounded (consider $A = F_n$ where $(F_n)_{n \geq 0}$ is a Følner sequence)

"The faster the isoperimetric profile tends to infinity, the more the group is amenable"

Rigidity result on quantitative orbit equivalence

Given $f, g\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\boldsymbol{f \preccurlyeq g}$ ("g dominates f") means that:

$\exists C > 0,\ f(x) \leq Cg(Cx)$ for sufficiently large positive real numbers x.

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Theorem (Delabie, Koivisto, Le Maître, Tessera 2023)

Let G and H be finitely generated groups.
Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing map such that $t \mapsto t/\varphi(t)$ is increasing.

If there exists a (φ, L^0)-integrable OE coupling from G to H, then

$$\varphi(j_{1,H}(x)) \preccurlyeq j_{1,G}(x).$$

Application of DKLMT's theorem for $\mathbb{Z}^d$

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It is sharp!

Theorem (Delabie, Koivisto, Le Maître, Tessera 2023)

For every $p < \frac{k}{k+\ell}$, there exists an $(\mathrm{L}^p, \mathrm{L}^0)$-integrable OE coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$.

Optimality results for $\mathbb{Z}^d$

Corollary (Delabie, Koivisto, Le Maître, Tessera 2023)

Existence of an $(\mathrm{L}^p, \mathrm{L}^0)$*-integrable OE coupling from* $\mathbb{Z}^{k+\ell}$ *to* $\mathbb{Z}^k$*:*

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What about $p = \frac{k}{k+\ell}$?

Theorem (C. 2025)

This threshold cannot be reached!

Wreath products, lamplighters

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Definition

A wreath product $\Lambda \wr H$ is a lamplighter if Λ is finite.

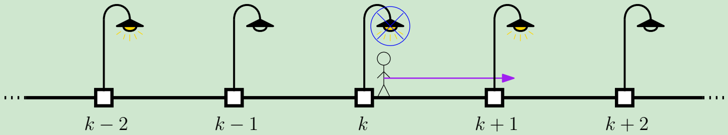
$$\Lambda \wr H := \left(\bigoplus_H \Lambda\right) \rtimes H$$

Example

$\Lambda = \mathbb{Z}/2\mathbb{Z}$, $H = \mathbb{Z}$.

$f \in \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \rightsquigarrow$ on the site $n \in \mathbb{Z}$, $\begin{cases} f(n) = 0 : & \text{light off} \\ f(n) = 1 : & \text{light on} \end{cases}$

$k \in \mathbb{Z} \rightsquigarrow$ position of the lamplighter.



Two generators: $([n \in \mathbb{Z} \mapsto 0 \in \mathbb{Z}/2\mathbb{Z}], 1)$ and $([n \in \mathbb{Z} \mapsto \mathbb{1}_{n=0} \in \mathbb{Z}/2\mathbb{Z}], 0)$.

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$\leadsto$ If there exists a (φ, L^0)-integrable OE coupling from $\Lambda \wr \mathbb{Z}$ to $\mathbb{Z}$, then $\varphi(x) \preccurlyeq \log x$.

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It is sharp !

Theorem (Delabie, Koivisto, Le Maître, Tessera 2023)

For every $p < 1$, there exists a $(\log^p, \mathrm{L}^0)$-integrable OE coupling from $\Lambda \wr \mathbb{Z}$ to $\mathbb{Z}$.

Optimality results for the lamplighter $\Lambda \wr \mathbb{Z}$

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Existence of a $(\log^p, \mathrm{L}^0)$*-integrable OE coupling from* $\Lambda \wr \mathbb{Z}$ *to* $\mathbb{Z}$*:*

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What about $p = 1$?

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Other constructions of OE couplings

Theorem (Delabie, Koivisto, Le Maître, Tessera 2023)

Let Λ be a finite group.
Let H and K be finitely generated groups.
Let $\varphi, \psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be increasing maps.

If there exists a (φ, ψ)-integrable OE coupling from H to K,
then there exists a (φ, ψ)-integrable OE coupling from $\Lambda \wr H$ to $\Lambda \wr K$.

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Using a result of **composition** of couplings, we can **iterate** these results and get a similar statement between

$$\underbrace{\Lambda \wr (\Lambda \wr (\ldots \Lambda \wr (\Lambda \wr}_{n \text{ times}} H) \ldots)) \text{ and } \underbrace{\Lambda \wr (\Lambda \wr (\ldots \Lambda \wr (\Lambda \wr}_{n \text{ times}} K) \ldots)).$$

Our results

Let's talk about our joint work with Vincent Dumoncel

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Our results

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- we consider **lampshuffler groups**
- we build **orbit equivalence** couplings between these groups
- we **quantify** the cocycles
- we compute the **isoperimetric profiles** to prove quantitative **optimality** of these couplings.

What is a lampshuffler ?

Recall:

Lamplighter: $\Lambda \wr H := (\bigoplus_H \Lambda) \rtimes H$.

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"Lamplighter-like" groups: lampshufflers

Definition

Given a group H, $\mathsf{Shuffler}(H) := \mathrm{FSym}(H) \rtimes H$

- $\mathrm{FSym}(H)$: set of permutations $\sigma \colon H \to H$ of finite support
  (i.e. $\{h \in H \mid \sigma(h) \neq h\}$ is finite),
  seen as relabellings of the elements of H: $h \in H$ carries the label $\sigma(h)$
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Orbit equivalence couplings between lampshufflers

$\mathsf{Shuffler}^{(0)}(H) = H$, $\mathsf{Shuffler}^{(n+1)}(H) = \mathsf{Shuffler}^{(n)}(\mathsf{Shuffler}(H))$

Theorem (C., Dumoncel 2025+)

Let H and K be finitely generated groups.
Let $\varphi, \psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be increasing maps.

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Theorem (C., Dumoncel 2025+)

For every $p < \frac{1}{k}$, there exists a $((\underbrace{\log \circ \ldots \circ \log}_{n \text{ times}})^p, \mathrm{L}^0)$-integrable OE coupling from $\mathsf{Shuffler}^{(n)}(\mathbb{Z}^k)$ to $\mathbb{Z}^d$.

Isoperimetric profiles of lampshufflers, optimality of our couplings

We use DKLMT's theorem to check that our couplings are quantitatively optimal. **But what are the isoperimetric profiles of lampshufflers ?**

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Theorem (Erschler, Zheng 2023)

$$j_{1,\mathsf{Shuffler}(\mathbb{Z}^k)}(x) \simeq \left(\frac{\log(x)}{\log(\log(x))}\right)^{1/k}$$

(holds true for a group H having polynomial growth of degree k)

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For other amenable groups H, bounds for $j_{1,\mathsf{Shuffler}(H)}$ have been found [Saloff-Coste - Zheng 2021, Erschler - Zheng 2023], **not sharp** in full generality.

For instance, $j_{1,\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^k)}$ was not known.

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H finitely generated amenable group such that

- $\forall C > 0,\ j_{1,H}(Cx) = O\left(j_{1,H}(x)\right);$

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Then $j_{1,\mathsf{Shuffler}(H)}(x) \simeq j_{1,H}(\log(x))$.

We also get the isoperimetric profiles of iterated lampshufflers $\mathsf{Shuffler}^{(\circ n)}(H)$.

Isoperimetric profiles of lampshufflers, optimality of our couplings

Theorem (C., Dumoncel 2025+)

H finitely generated amenable group such that

- $\forall C > 0,\ j_{1,H}(Cx) = O\left(j_{1,H}(x)\right)$;
- $j_{1,H}\left(\frac{\log(x)}{\log(\log(x))}\right) \simeq j_{1,H}(\log(x))$.

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We don't have any example of a group H which doesn't have polynomial growth and which doesn't satisfy the assumptions of our theorem.

We can then check that our OE couplings are quantitatively optimal!

Towards halo products

We also investigate OE and isoperimetric profiles for other "lamplighter-like" groups: **halo products**.

Halo products have been introduced by Genevois and Tessera as a natural generalisation of lamplighters and lampshufflers.
They proved many results on quasi-isometry between these groups.

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\begin{publicity}

Examples of halo products (with fancy names!):
lampjugglers, lampdesigners, lampcloners, lampbraiders, ...

\end{publicity}

Thank you for listening !