Quantitative orbit equivalence for p.m.p. $\mathbb{Z}$-actions, an overview, and more details on flexibility results provided by odomutants

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These notes first aim at giving an overview of all the results on quantitative orbit equivalent in the context of actions of the group $\mathbb{Z}$, which reduces to the data of an invertible transformation representing a generator of $\mathbb{Z}$.

Secondly, we will explain with few details recent flexibility results using a class of interesting systems, called odomutants, introduced by the author (the paper is not yet online).

The basic definitions and properties of ergodic theory are given in Appendix A

Contents

1 Quantitative orbit equivalence: first definitions

The probability space $(X, \mathcal{A}, \mu)$ is assumed to be standard and atomless. Such a space is always isomorphic to $([0,1], \mathcal{B}([0,1]), \mathrm{Leb})$. We consider maps $T\colon X \to X$ acting on this space and which are bijective, bimeasurable and **probability measure-preserving** (**p.m.p.**), meaning that $\mu(T^{-1}(A)) = \mu(A)$ for all measurable sets $A \subset X$, and the set of these transformations is denoted by $\mathrm{Aut}(X, \mathcal{A}, \mu)$, or simply $\mathbf{Aut}(\boldsymbol{X}, \boldsymbol{\mu})$, two such maps being identified if they coincide on a measurable set of full measure. In this paper, elements of $\mathrm{Aut}(X, \mu)$ are called **transformations** or (**dynamical**) **systems**. One of the main goals in ergodic theory is to classify systems up to conjugacy.

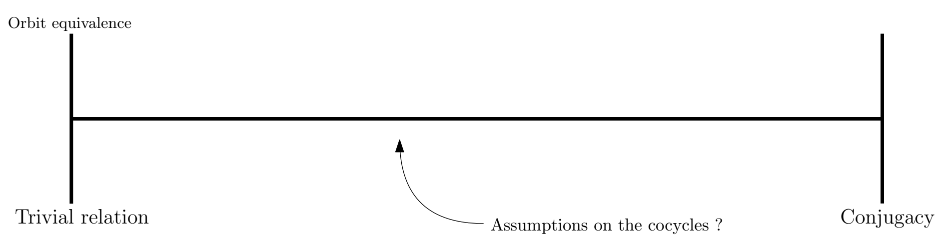
Definition 1.1. Two transformations $T \in \mathrm{Aut}(X, \mu)$ and $S \in \mathrm{Aut}(Y, \nu)$ are **conjugate** if there exists a bimeasurable bijection $\Psi\colon X \to Y$ such that $\Psi_\star \mu = \nu$ and $\Psi \circ T = S \circ \Psi$ almost everywhere.

Some classes of transformations have been classified up to conjugacy, the two examples to keep in mind are the following. By Ornstein [Orn70], entropy is a total invariant of conjugacy among Bernoulli shifts. Moreover Halmos and von Neumann [HVN42] prove that two ergodic systems with discrete spectrums are conjugate if and only if they have equal point spectrums. In full generality, the problem of conjugacy is intractable and we look for weaker relations, such as orbit equivalence.

Definition 1.2. Two transformations $S \in \mathrm{Aut}(X, \mu)$ and $T \in \mathrm{Aut}(Y, \nu)$ are **orbit equivalent** if there exists a bimeasurable bijection $\Psi\colon X \to Y$ satisfying $\Psi_\star \mu = \nu$, such that $\mathrm{Orb}_S(x) = \mathrm{Orb}_{\Psi^{-1}T\Psi}(x)$ for almost every $x \in X$. The map Ψ is called an **orbit equivalence** between S and T.

Notice that orbit equivalence and ergodicity are two properties related to the orbits. It is easy to see that orbit equivalence preserves ergodicity. Actually, Dye's theorem states that this equivalence relation is too weak among ergodic systems.

Theorem 1.3 (Dye [Dye59])**.** *Two ergodic transformations T in $\mathrm{Aut}(X, \mu)$ and $S \in \mathrm{Aut}(Y, \nu)$ are orbit equivalent.*



Since orbit equivalence cannot distinguish between transformations, we have to strengthen the definition of orbit equivalence in order to capture some dynamical properties. To this end, we consider functions, called the cocycles, which more precisely describe the equality of the orbits.

Definition 1.4. In the case of aperiodic transformations, we can define the **cocycles** associated to an orbit equivalence. These are measurable functions $c_S\colon X \to \mathbb{Z}$ and $c_T\colon Y \to \mathbb{Z}$ defined almost everywhere by

$$Sx = \Psi^{-1} T^{c_S(x)} \Psi(x) \text{ and } Ty = \Psi S^{c_T(y)} \Psi^{-1}(y)$$

($c_S(x)$ and $c_T(y)$ are uniquely defined by aperiodicity).

"Aperiodic" means that almost every orbit is infinite, or equivalently that for almost every x, $T^n(x) \neq x$ for every nonzero integer n. Ergodicity implies aperiodicity.

As illustrated in the Figure 1, given an orbit equivalence between two transformations S and T, the cocycles encode the distortion of the orbits to move from the dynamic of T to the dynamic of S. So we expect that two orbit equivalent transformations have similar dynamical properties if we add strong enough restrictions on these cocycles (⚠ we warn the reader that the cocycles depend on the orbit equivalence we consider).

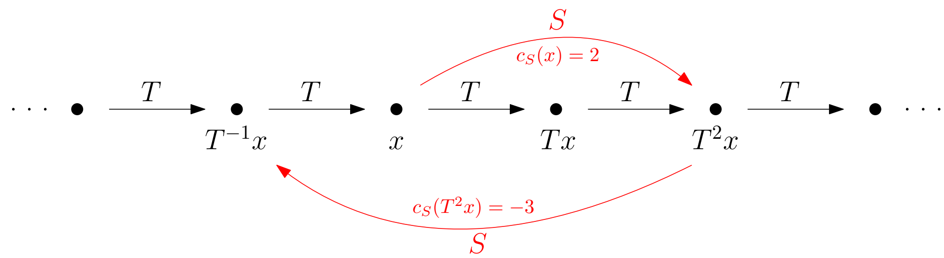


Figure 1: Orbit equivalence between aperiodic transformations, when $\Psi = id_X$.

Definition 1.5. Given a map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, a measurable function $f\colon X \to \mathbb{Z}$ is said to be **φ-integrable** if

$$\int_X \varphi(|f(x)|)\mathrm{d}\mu(x) < +\infty.$$

Two transformations in $\mathrm{Aut}(X,\mu)$ are said to be **φ-integrably orbit equivalent** if there exists an orbit equivalence between them whose associated cocycles are φ-integrable. The notion of L^p **orbit equivalence** refers to the map $\varphi\colon x \to x^p$, and a $\mathrm{L}^{<p}$ **orbit equivalence** is by definition an orbit equivalence which is L^q for all $q < p$.
Another form of quantitative orbit equivalence is Shannon orbit equivalence. We say that a measurable function $f\colon X \to \mathbb{Z}$ is **Shannon** if the associated partition $\{f^{-1}(n) \mid n \in \mathbb{Z}\}$ of X has finite entropy, namely

$$-\sum_{n\in\mathbb{Z}} \mu(f^{-1}(n)) \log \mu(f^{-1}(n)) < +\infty.$$

Two transformations in $\mathrm{Aut}(X,\mu)$ are **Shannon orbit equivalent** if there exists an orbit equivalence between them whose associated cocycles are Shannon.

For example, integrability is exactly φ-integrability when φ is non-zero and linear, and a weaker quantification on cocycles is the notion of φ-integrability for a sublinear map φ, meaning that $\lim_{t\to+\infty} \varphi(t)/t = 0$.

When φ is non-decreasing, then φ-integrability of a $\mathbb{Z}$-valued function f gives information on its tail, using Markov's inequality:

$$\mu\left(|f(x)| > n\right) = \mu\left(\varphi(|f(x)|) > \varphi(n)\right) \leqslant \frac{\int_X \varphi(|f(x)|)\mathrm{d}\mu(x)}{\varphi(n)}.$$

On the contrary to the statistical information provided by φ-integrability, Shannon property quantifies the uncertainty of the value of $f(x)$, if x is random with respect to μ (see Section A.3 in the appendix).

⚠ φ-integrably orbit equivalence and Shannon orbit equivalence are not equivalence relation a priori (except when φ is at least linear, by Belinskaya's theorem, see Section 2.1)

2 Overview : rigidity/flexibility results

As explained in the last section, we want to weaken the difficult problem of conjugacy. However orbit equivalence is a trivial relation so it is not an interesting theory. Now quantitative forms of orbit equivalence are strenghtenings of orbit equivalence and we want to know if they are still far from the difficult problem of conjugacy. Then we will be interested in dynamical properties preserved under these relations, or flexbility results.

2.1 Conjugacy/flip-conjugacy

- When φ is a nonnull linear map (we talk about integrable orbit equivalence), we recover a conjugacy problem.

Theorem 2.1 (Belinskaya [Bel69])**.** *Let S and T be transformation in $\mathrm{Aut}(X,\mu)$. If there exists an orbit equivalence between them such that one the cocycles is integrable, then S and T are flip-conjugate, meaning that S is conjugate to T or to T^{-1}.*

- When φ is sublinear, it is natural to wonder whether φ-integrable orbit equivalence boils down to flip-conjugacy.
  - Let us start with an asymmetric result (as for Belinskaya's theorem) when we only assume φ-integrability on one cocycle.

    Theorem 2.2 (Carderi, Joseph, Le Maître, Tessera [CJLMT23])**.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and let $S \in \mathrm{Aut}(X,\mu)$ be ergodic. There exists $T \in \mathrm{Aut}(X,\mu)$ and an orbit equivalence between S and T such that the cocycle c_T is φ-integrable but T and S are not flip-conjugate.*

  - What about adding restrictions on both cocycles ?

    Theorem 2.3 (Carderi, Joseph, Le Maître, Tessera [CJLMT23])**.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and let $S \in \mathrm{Aut}(X,\mu)$ be ergodic. Assume that there exists an integer $n \geqslant 2$ such that S^n is ergodic. Then there exists $T \in \mathrm{Aut}(X,\mu)$ such that S and T are φ-integrably orbit equivalent but not flip-conjugate.*

The proof of the last theorem is constructive and the system T is built so that T^n is not ergodic. Theorem 2.2 is a direct corollary of Theorem 2.3 when S admits an integer $n \geqslant 2$ such that S^n is ergodic. For the systems S which do not satisfy this assumption, the authors used a G_δ argument to prove that we can find a weakly mixing system T, and then Theorem 2.2 follows from the fact that S is not weakly mixing.

  - Examples of systems S Theorem 2.3 does not apply for are some odometers (see Remark 3.1). In these notes, we will explain the following theorem.

    Theorem A (C.)**.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and let $S \in \mathrm{Aut}(X,\mu)$ be an odometer. Then there exists $T \in \mathrm{Aut}(X,\mu)$ such that S and T are φ-integrably orbit equivalent but not flip-conjugate.*

    The system T that we build is an odomutant associated to S. We will later introduce these systems which forms a source of counter-example for many flexibility results.

- What about Shannon orbit equivalence ?

  Theorem 2.4 (Carderi, Joseph, Le Maître, Tessera [CJLMT23])**.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map satisfying $\log(t) = O(\varphi(t))$ as t goes to ∞. If a $\mathbb{Z}$-valued function f is φ-integrable, then it is Shannon.*

  As a corollary, when φ satisfies the assumption of the theorem, φ-integrable orbit equivalence implies Shannon orbit equivalence, and Theorems 2.2 and 2.3 hold true in the case of Shannon orbit equivalence.

- The following result is the first example of famous dynamical systems related by these quantitative forms of quantitative orbit equivalence but which are not flip-conjugate.

  Theorem 2.5 (Kerr, Li [KL24])**.** *Every odometer is Shannon orbit equivalent to the universal odometer.*

In recent years, odometers have played a crucial for explicit constructions of orbit equivalence, thanks to their combinatorial structure, that is what Kerr and Li used to prove their theorem. We generalized their construction in [Cor24] to rank-one systems which almost have the same structure as odometers but form a class of systems with various dynamical properties (whereas the subclass of odometers is not rich enough), thus providing flexibility results (see Theorems 2.9, 2.10, 2.11 and 2.12). In [DKLMT22], Delabie, Koivisto, Le Maître and Tessera built concrete orbit equivalences between actions of amenable groups using Følner tilings, which turn out to be some kind of generalizations of odometers for more general group actions. Finally, we use the combinatorial structure of odometers to build our odomutants and get flexibility results. We will introduce later odometers, their classification up to flip-conjugacy (and what it means to be universal) and we will explain their structures.

Let us now see the behaviour of dynamical properties under these quantitative forms of orbit equivalence.

2.2 Ergodicity

As explained in the last section, ergodicity and orbit equivalence are two relations related to orbits and we have the followings: given $S, T \in \operatorname{Aut}(X, \mu)$, with S ergodic,

- if S and T are orbit equivalent, then T is ergodic;
- the converse is true by Dye's theorem : if T is ergodic, then S and T are orbit equivalent.

2.3 Mixing properties

Weak mixing property is not preserved under the quantitative forms of orbit equivalence associated to a sublinear map, or under Shannon orbit equivalence.

- If S is weakly mixing, then all its powers are weakly mixing, and since weak mixing property implies ergodicity, all the powers of S are ergodic. Since the counter-example T in Theorem 2.3 is built as a system admitting a power which is not ergodic, we deduce the following.

  Corollary 2.6. *Given a sublinear map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, every weakly mixing system is φ-integrably orbit equivalent to a non weakly mixing system.*

- Moreover, the proof of Theorem 2.2 shows (see the paragraph after Theorem 2.3):

  Corollary 2.7. *Given a sublinear map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, every non weakly mixing system S is φ-integrably orbit equivalent to a weakly mixing system T such that c_T is φ-integrable.*

- A recent preprint proves a stronger result in the context of Shannon orbit equivalence.

  Theorem 2.8 (O'Quinn [O'Q24])**.** *Every system in $\operatorname{Aut}(X, \mu)$ is Shannon orbit equivalent to a weakly mixing system.*

- We found concrete a concrete example of systems which are Shannon orbit equivalent, one is weakly mixing and the other is not.

  Theorem 2.9 (C. [Cor24])**.** *If $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\varphi(x) = o(x^{1/3})$, then the universal odometer and Chacon map are φ-integrably orbit equivalent.*

Historically, Chacon map is the first example of weakly mixing system which is not strongly mixing. It is defined with a cutting-and-stacking construction with only one tower at each step, such a system is called a *rank-one system*. Chacon map was actually the first system defined this way and it opens to the study of rank-one systems.

We can neither find rigidity results for strong mixing property.

- Indeed, Corollary 2.6 can also be stated in terms of strong mixing property, by the same arguments.
- Moreover we found a concrete example:

  Theorem 2.10 (C. [Cor24])**.** *If $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\varphi(x) = o(x^{1/3})$, then the universal odometer is φ-integrably orbit equivalent to a strongly mixing rank-one system.*

2.4 Eigenvalues, point spectrum

Again, we only have flexibility results about eigenvalues.

- First, Theorem 2.5 provides two Shannon orbit equivalent systems with different point spectrums (see Sections 3.1.2 and 3.1.3). Moreover, Theorem 2.3 provides two systems which do not have the same point spectrum. Indeed, given a prime number n and an ergodic system $T \in \operatorname{Aut}(X, \mu)$, if T^n is not ergodic, then T has $\exp\left(\frac{2i\pi}{n}\right)$ as an eigenvalue, and the converse is true.

- We also provide concrete examples with irrational rotations. Note that the irrational rotation of angle $\theta \in \mathbb{R}\backslash\mathbb{Q}$ has $\exp(2i\pi\theta)$ as an eigenvalue, whereas the eigenvalues of odometers are all rational (rational means that it has the form $\exp(2i\pi\alpha)$ where α is rational).

> **Theorem 2.11** (C. [Cor24])**.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map satisfying $\varphi(x) = o(x^{1/3})$. There exists a uncountable dense subset Θ of $\mathbb{R}\backslash\mathbb{Q}$ such that for every $\theta \in \Theta$, the universal odometer is φ-integrably orbit equivalent to the irrational rotation of angle θ.*

- The previous statement do not deal with all the irrational number θ a priori. For the remaining θ, we have the following weaker statement.

> **Theorem 2.12** (C. [Cor24])**.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map satisfying $\varphi(x) = o(x^{1/3})$. For every irrational number θ, the universal odometer is φ-integrably orbit equivalent to a system which has* $\exp(2i\pi\theta)$ *as an eigenvalue.*

2.5 Entropy

Fortunately, some quantitative forms of orbit equivalence (not only integrable orbit equivalence) are not trivial relations like orbit equivalence itself.

- Entropy is an invariant of Shannon orbit equivalence:

> **Theorem 2.13** (Kerr, Li [KL24])**.** *Let $S, T \in \mathrm{Aut}(X, \mu)$. If S and T are Shannon orbit equivalent, then $\mathrm{h}_\mu(S) = \mathrm{h}_\mu(T)$.*

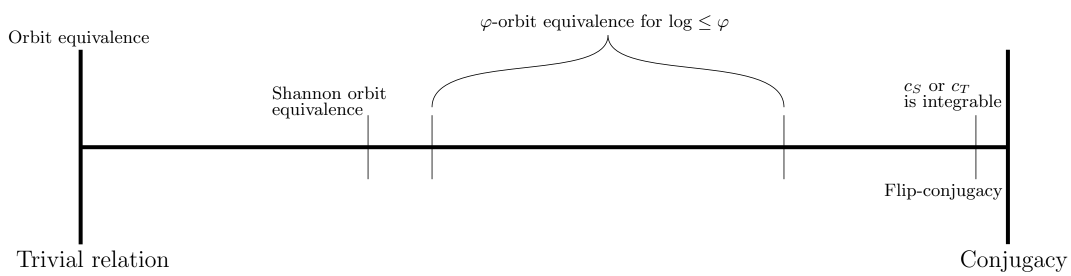


Figure 2: Here is a schematic view of the interplay between the relations on ergodic bijections we have seen so far.

- From Theorem 2.4, we deduce:

> **Corollary 2.14.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map satisfying $\log(t) = O(\varphi(t))$ as t goes to ∞. Let $S, T \in \mathrm{Aut}(X, \mu)$. If S and T are φ-integrably orbit equivalent, then $\mathrm{h}_\mu(S) = \mathrm{h}_\mu(T)$.*

- This result is optimal.

> **Theorem 2.15** (C.)**.** *Let α be a positive real number or $+\infty$. There exist $S, T \in \mathrm{Aut}(X, \mu)$ such that:*
>
> 1. $\mathrm{h}_\mu(S) = 0$;
> 2. $\mathrm{h}_\mu(T) = \alpha$;
> 3. *there exists an orbit equivalence between S and T, which is φ_m-integrable for all integers $m \geqslant 0$,*
>
> *where φ_m denotes the map $t \to \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log \circ \ldots \circ \log$ (m times).*

Outlines of the proof of Theorem 2.15:

– In the proof, S is an odometer and T an associated odomutant. We can also describe a large class of odometers for which the theorem applies.

– Topological entropy is easier to use in this context. If T is uniquely ergodic, we can connect $\mathrm{h}_{\mathrm{top}}(T)$ with $\mathrm{h}_\mu(T)$, via the variational principle.

– For $\mathrm{h}_{\mathrm{top}}(T)$ to be well-defined, T has to be a topological system. It will be possible, with some assumptions, to extend odomutants to minimal homeomorphisms on the Cantor set X. The orbit equivalence with the odometer becomes a strong orbit equivalence.

– If T is strongly orbit equivalent to a uniquely ergodic system S, then T is uniquely ergodic. Therefore, $\mathrm{h}_{\mathrm{top}}(T) = \mathrm{h}_\mu(T)$.

– It remains to find the parameters (q_n) and the permutations so that $\mathrm{h}_{\mathrm{top}}(T) = \alpha$ and the orbit equivalence is almost log-integrable.

Hence the more general statement (supernatural numbers are introduced in Section 3.1.3):

Theorem 2.16 (C.). *Let α be either a positive real number or $+\infty$. Let S be an odometer whose associated supernatural number $\prod_{p\in\Pi} p^{k_p}$ satisfies the following property: there exists a prime number $p_\star$ such that $k_{p_\star} = +\infty$. Then there exists a Cantor minimal homeomorphism T such that*

1. $\mathrm{h}_{\mathrm{top}}(T) = \alpha$;
2. *there exists a strong orbit equivalence between S and T, which is φ_m-integrable for all integers $m \geqslant 0$,*

where φ_m denotes the map $t \to \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log \circ \ldots \circ \log$ (m times).

Examples of odometers S to which this theorem applies are the dyadic odometer, more generally the p-odometer for every prime number p, or the universal odometer.

We will detail the proof of the following weaker version:

Theorem B (C.). *Let S be the universal odometer. Then there exists a Cantor minimal homeomorphism T such that*

1. $\mathrm{h}_{\mathrm{top}}(T) > 0$;
2. *there exists a strong orbit equivalence between S and T, which is φ_m-integrable for all integers $m \geqslant 0$,*

where φ_m denotes the map $t \to \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log \circ \ldots \circ \log$ (m times).

- Let us point out that Theorem 2.16 generalizes a famous theorem of Boyle and Handelman.

Theorem 2.17 (Boyle, Handelman [BH94]). *Let α be either a positive real number or $+\infty$. Let S be the dyadic odometer. Then there exists a Cantor minimal homeomorphism T such that*

1. $\mathrm{h}_{\mathrm{top}}(T) = \alpha$;
2. *S and T are strongly orbit equivalent.*

Boyle and Handelman use

– Bratteli diagrams: combinatorial objects encoding the cutting-and-stacking construction of the Cantor minimal homeomorphisms, see [HPS92];

– and the dimension groups: algebraic tools, provided by Bratteli diagrams, which form complete invariants of strong orbit equivalence, see [GPS95];

to prove that the system T that they build is strongly orbit equivalent to S. In particular, the orbit equivalence is established in an **abstract** way, whereas the orbit equivalence between an odometer and our odomutants is very concrete and we can also quantify the cocycles !

Actually, it turns out that the dynamical systems T built by Boyle and Handelman (via their Bratteli diagrams) are... odomutants! The work of Boyle and Handelman was not the starting point of our study of this new class of systems (see Section 2.6).

2.6 Loose Bernoullicity property

A well-known equivalence relation, called *even Kakutani equivalence*, preserves entropy. We may wonder if there exists a connexion with the quantitative forms of orbit equivalence.

- We give a partial answer, again using odomutants.

> **Theorem C** (C.). *There exists a system $T \in \mathrm{Aut}(X, \mu)$ which is $L^{<1/2}$ orbit equivalent (in particular Shannon orbit equivalent) to the dyadic odometer but not evenly Kakutani equivalent to it.*

As for the conjugacy problem, there exists a class of systems for which the problem of even Kakutani equivalence (and also Kakutani equivalence) is well-understood: the class of loosely Bernoulli systems (containing odometers, Bernoulli shifts, etc). It is closed under Kakutani equivalence and the entropy is a total invariant of even Kakutani equivalence. To prove Theorem C, the system T that we build is an odomutant which is not loosely Bernoulli. This is an example of zero-entropy and non loosely Bernoully system built by Feldman [Fel76] and its interesting structure, closed to odometers, led us to find a formal definition of such a system and to open the study of odomutants.

Question 2.18. Does even Kakutani equivalence imply some quantitative form of orbit equivalence ?

3 Towards odomutants

Topological entropy and loose Bernoulli property are properties related to words, and odometers are systems which do not produce a lot of words with respect to partitions, that is the reason why they have zero entropy and are loosely Bernoulli. To prove Theorems B and C, the goal is to build systems orbit equivalent to odometers and which produce a lot of words in such a way they have positive entropy and are not loosely Bernoully.

3.1 Odometers

3.1.1 Adding machine on the Cantor space

Given integers $q_0, q_1, q_2, \ldots$ greater than or equal to 2, let us consider the Cantor space

$$X \coloneqq \prod_{n \geqslant 0} \{0, 1, \ldots, q_n - 1\},$$

endowed with the infinite product topology and the associated Borel σ-algebra. The **odometer** on X is the adding machine $S\colon X \to X$, defined for every $x \in X$ by

$$Sx = \begin{cases} (0, \ldots, 0, 1 + x_i, x_{i+1}, \ldots) & \text{if } i \coloneqq \min\{j \geqslant 0 \mid x_j \neq q_j - 1\} \text{ is finite} \\ (0, 0, 0, \ldots) & \text{if } x = (q_0 - 1, q_1 - 1, q_2 - 1, \ldots) \end{cases}.$$

In other words, S is the addition by $(1, 0, 0, \ldots)$ with carry over to the right.

An odometer is more generally a system which is conjugate to S for some choice of integers q_n. In this paper, we only consider this concrete example with the adding machine and we refer to it as "the odometer on $\prod_{n \geqslant 0} \{0, 1, \ldots, q_n - 1\}$".

Let us introduce the **cylinders of length** k, or k**-cylinders**,

$$[x_0, \ldots, x_{k-1}]_k \coloneqq \left\{ (y_n)_{n \geqslant 0} \in \prod_{n \geqslant 0} \{0, 1, \ldots, q_n - 1\} \;\middle|\; y_0 = x_0, \ldots, y_{k-1} = x_{k-1} \right\}.$$

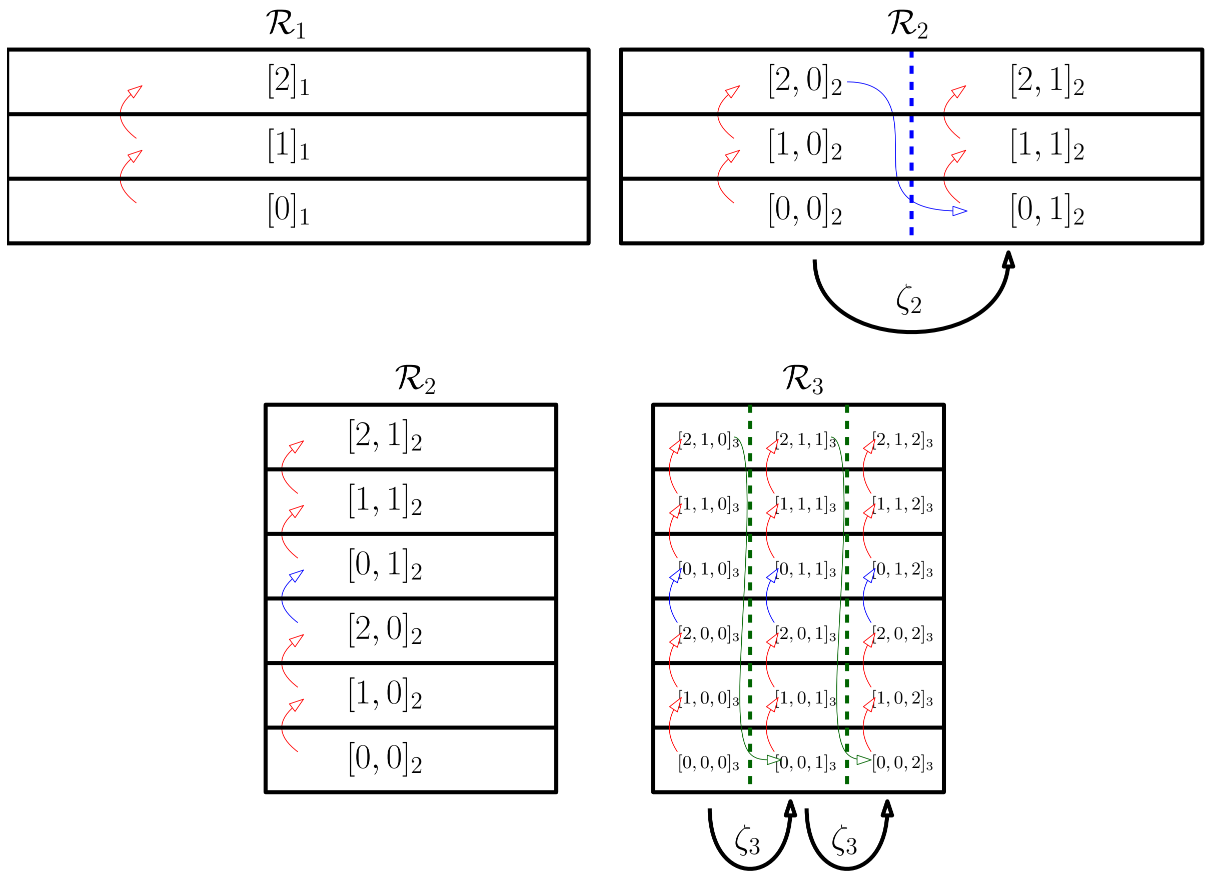


Figure 3: Example of odometer with $q_0 = 3$, $q_1 = 2$, $q_2 = 3$ (so $h_1 = 3$, $h_2 = 6$, $h_3 = 18$).

We also use the symbol $\bullet$ when we do not want to fix the value at some coordinate. For instance, $[x_0, \bullet, x_2]_3$ denotes the set of sequences $(y_n)_{n\geqslant 0}$ satisfying $y_0 = x_0$ and $y_2 = x_2$. By convention, the 0-cylinder is X. We can also set a partially defined map

$$\zeta_n \colon X \backslash [q_0 - 1, \dots, q_{n-1} - 1]_n \to X \backslash [0, \dots, 0]_n$$

which is the addition by

$$(\underbrace{0, \dots, 0}_{n-1 \text{ times}}, 1, 0, 0, \dots).$$

For example, S and ζ_1 coincide on $X \backslash [q_0 - 1]_1$. As illustrated in Figure 3, the cylinders and the maps ζ_n offer a very interesting combinatorial structure with successive nested towers $\mathcal{R}_1, \mathcal{R}_2, \dots$.[1]

From $(q_n)_{n\geqslant 0}$, a new sequence $(h_n)_{n\geqslant 1}$ is defined by

$$\forall n \geqslant 1,\ h_n \coloneqq q_0 q_1 \dots q_{n-1}.$$

The integer h_n is the height of the tower $\mathcal{R}_n$ (see Figure 3). By convention, we set $h_0 \coloneqq 1$, the height of the tower $\mathcal{R}_0 \coloneqq (X)$ with a single level.

As a topological system, S is a Cantor minimal homeomorphism. As a measure-theoretic system, S is uniquely ergodic and its only invariant measure is the product $\mu \coloneqq \bigotimes_{n\geqslant 0} \mu_n$ where μ_n is the uniform distribution on $\{0, 1, \dots, q_n - 1\}$.

3.1.2 Point spectrum

The odometer S on $\prod_{n\geqslant 0} \{0, 1, \dots, q_n - 1\}$ has discrete spectrum and its point spectrum is equal to

$$\left\{ \exp\left(\frac{2i\pi k}{h_n}\right) \mid n \geqslant 1, 0 \leqslant k \leqslant h_n - 1 \right\}$$

[1] This kind of construction that we see in Figure 3 is called a cutting-and-stacking construction.

(see Example A.10).

3.1.3 Classification up to flip-conjugacy

Let us now explain the classification of odometers up to conjugacy (and even flip-conjugacy).

Let Π denote the set of prime numbers. A **supernatural number** is a formal product of the form $\prod_{p\in\Pi} p^{k_p}$, with $k_p \in \mathbb{N} \cup \{+\infty\}$.

Given a prime number $p \in \Pi$, denote by $\nu_p(k)$ the p-adic valuation of a positive integer k. To every odometer defined with integers $q_0, q_1, \ldots$, we associate a supernatural number $\prod_{p\in\Pi} p^{k_p}$ defined by

$$k_p := \sum_{n\geqslant 0} \nu_p(q_n).$$

As a consequence of Section 3.1.2 and Halmos-von Neumann Theorem, the supernatural number $\prod_{p\in\Pi} p^{k_p}$ forms a total invariant of measure-theoretic conjugacy in the class of odometers. If $k_p = \infty$ for every prime number p, then the odometer is said to be **universal**. Given a prime number p, the **p-odometer** is the odometer such that $k_p = \infty$ and $k_q = 0$ for every $q \in \Pi\backslash\{p\}$. In the case $p = 2$, it is also called the **dyadic** odometer.

Remark 3.1. Given the odometer S on $\prod\{0, \ldots, q_n - 1\}$ and a positive integer k, the system S^k is not ergodic if and only if k divides h_n when n is large enough. So the odometers for which Theorem 2.3 does not apply are the ones whose associated supernatural numbers satisfy $k_p > 0$ for every prime number p.

3.1.4 Coalescence

Definition 3.2. We say that S is a factor of T, or T is an extension of S, if there exists a measurable map $\Psi\colon X \to Y$ which is onto and such that $\Psi_\star\nu = \mu$ and $S \circ \Psi = \Psi \circ T$ almost everywhere. The map Ψ is called a factor map from T to S.

Definition 3.3. A transformation $S \in \mathrm{Aut}(X, \mu)$ is coalescent if every system $T \in \mathrm{Aut}(X, \mu)$ which is isomorphic to S satisfies the following: every factor map from T to S is an isomorphism.

Every odometer is coalescent. This fact is proven in [HP68] and [New71]. In these articles, one proves that more general systems are coalescent and the phenomenon can be generalized in the context of group actions (see [IT16]). Here we give a short proof for ergodic systems with discrete spectrum.

Theorem 3.4. *Every ergodic system with discrete spectrum is coalescent.*

Proof of Theoreme 3.4. Let $S \in \mathrm{Aut}(X, \mu)$ be an ergodic system with discrete spectrum, $T \in \mathrm{Aut}(X, \mu)$ an extension of S and $\Psi\colon X \to X$ a factor map from T to S. Let us denote by E_S (resp. E_T) the set of all the eigenfunctions of S (resp. T). It is easy to check that if f_λ is an eigenfunction of S, associated to the eigenvalue $\lambda \in \mathrm{Sp}(S)$, then λ is also an eigenvalue of T and $f_\lambda \circ \Psi$ is an associated eigenfunction. Therefore, if S and T are conjugate, then ergodicity implies that eigenspaces have dimension at most 1 and we get $E_T = \{f \circ \Psi \mid f \in E_S\}$. This implies

$$\mathrm{L}^2(X, \mu) = \{f \circ \Psi \mid f \in \mathrm{L}^2(X, \mu)\}$$

since they have discrete spectrum. Hence Ψ is an isomorphism. □

For the proof of Theorem A, the systems that we will consider will be an odometer S and an associated odomutant T (the odomutants are introduced in Section 4). Since the odomutants are extensions of their associated odometer and since we explicitely know a factor map ψ between them (see Proposition 4.3), Theorem 3.4 will ensure that we will not build an orbit equivalence between flip-conjugate systems if ψ is not invertible.

3.1.5 Other properties

Odometers have zero measure-theoretic and topological entropies, and are loosely Bernoulli (see Appendix A for definitions). This is the consequence of the "poor" dynamic of the odometers (recall that an odometer act on cylinders of the same length in a cyclic way, and that cylinders generate the σ-algebra). The idea behind the construction of odomutants is to enrich the dynamic, namely to diversify the language with respect to some partitions.

⚠Informally, note that a loosely Bernoulli system does not necessarily have a "poor" dynamic. For instance, Bernoulli shifts are loosely Bernoulli.

Proposition 3.5. *Odometers have zero measure-theoretic and topological entropies.*

Proof of Proposition 3.5. Let S be an odometer. The equality $\mathrm{h}_\mu(S) = \mathrm{h}_{\mathrm{top}}(S)$ follows from unique ergodicity and the variational principle. Let $\mathcal{P}(k)$ be the partition given by the cylinders of length k. The odometer S acts as a cyclic permutation on the elements of $\mathcal{P}(k)$, so the sequence $((\mathcal{P}(k))_0^{n-1})_{n\geqslant 1}$ of partitions is stationary and we have $\mathrm{h}_\mu(S, \mathcal{P}(k)) = 0$. Since the product space $X = \prod_{n\geqslant 0}\{0, 1, \dots, q_n - 1\}$ is equipped with the σ-algebra generated by the cylinders, the sequence $(\mathcal{P}(k))_{k\geqslant 0}$ increases to the σ-algebra of X, so we have $\mathrm{h}_\mu(S, \mathcal{P}(k)) \underset{k\to+\infty}{\to} \mathrm{h}_\mu(S)$ by Proposition A.17, and we get $\mathrm{h}_\mu(S) = 0$. □

Proposition 3.6. *Odometers are loosely Bernoulli.*[a]

[a] More generally, rank-one systems are loosely Bernoulli, this is proven by Ornstein, Rudolph and Weiss [ORW82] (see Lemma 8.1) and we present their proof in the special case of odometers.

Proof of Proposition 3.6. Let S be an odometer, associated to the integers $q_0, q_1, \dots$, let $\mathcal{P}(k)$ be the partition given by the cylinders of length $k+1$. We denote by W the word $\left(S^i([0, \dots, 0]_k)\right)_{0\leqslant i\leqslant h_k-1} \in (\mathcal{P}(k))^{\{0,\dots,h_k-1\}}$, this is the enumeration of the $(k+1)$-cylinders, with the order given by the dynamic of S.

It is easy to check that $(S, \mathcal{P}(k))$ is loosely Bernoulli, since $[\mathcal{P}(k)]_{1,N}(x)$ (the word giving the future) is completely determined by $\mathcal{P}(k)(x)$ (so by $[\mathcal{P}(k)]_{-M,0}(x)$ giving the past) and the possible words in the future are f-close when N is large enough (they share a subword built as a concatenation of many copies W).

Since the sequence $(\mathcal{P}(k))_{k\geqslant 0}$ increases to the σ-algebra of X, we get that $(S, \mathcal{P})$ is loosely Bernoulli for every finite partition $\mathcal{P}$, so S is loosely Bernoulli. □

3.2 More complex cutting-and-stacking construction to get various dynamical properties

As explained before, we want to mutate odometers so that their language with respect to some partitions (e.g. partition in 1-cylinders) get richer. More precisely, we want these systems to either have more words (in order to get positive entropy) or to have less predictable laws for the future conditionally to a past (in order not to be loosely Bernoulli).

To this end, we have to revisit the cutting-and-stacking process defining the odometers, more precisely the way we connect the columns at each step of the construction. The idea is the following: each column is cutted in subcolumns, so that a tower at some step is divided in subtowers, and the way we connect the subcolumns of a same subtower does not necessarily correspond to the dynamic we would have with an odometer. We illustrate it in Figure 4.

4 Formal definition of odomutants, first properties

4.1 Definition

Let $X := \prod_{n\geqslant 0}\{0, 1, \dots, q_n - 1\}$ with integers $q_n \geqslant 2$, and let us recall the notation $h_n := q_0 \dots q_{n-1}$. The space X is endowed with the infinite product topology and we denote by μ the product of the uniform distributions on each

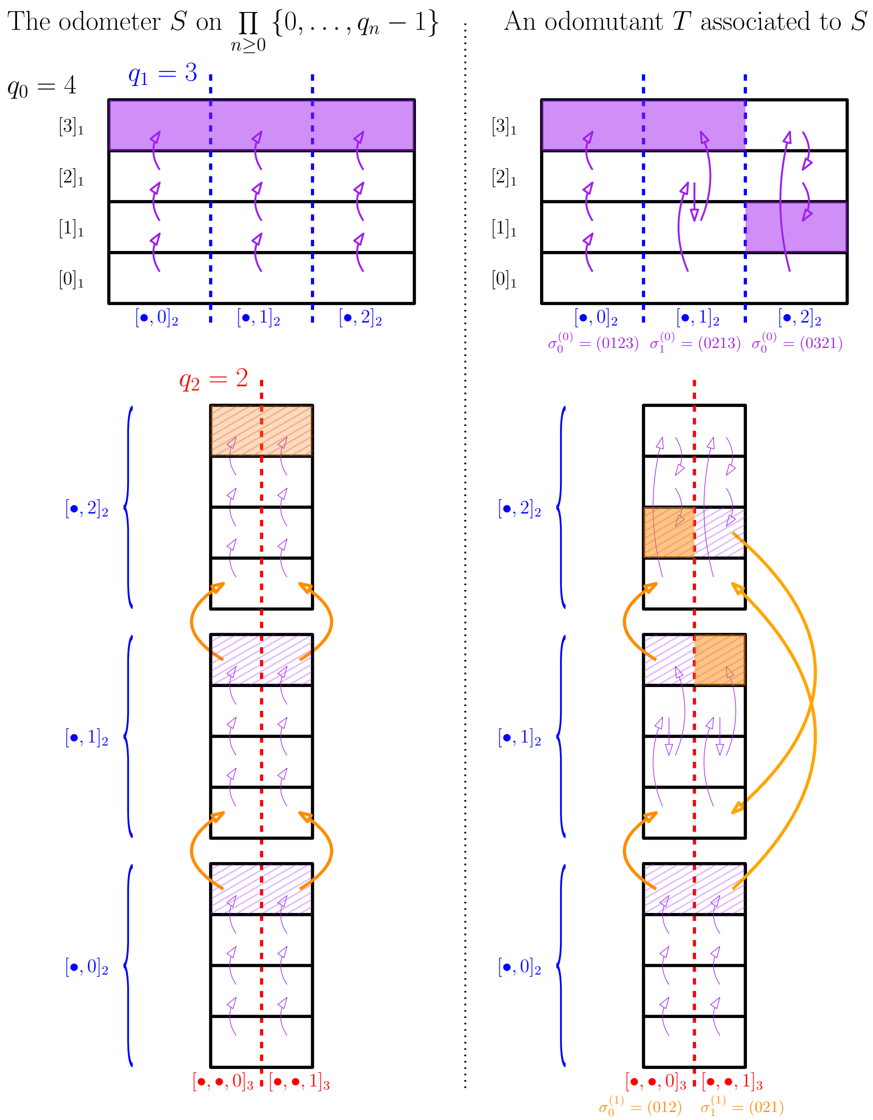


Figure 4: Example of the first two steps in the construction of an odometer (on the left) and an associated odomutant (on the right). For a permutation σ of the set $\{0,\dots,k-1\}$, the notation $\sigma = (i_0 \dots i_{k-1})$ means that σ is defined by $\sigma(j) = i_j$ for every $j \in \{0,\dots,k-1\}$. The area coloured in purple (resp. orange) is the subset on which S and T are not yet defined at the end of the first step (resp. second step).
For the odometer, the 1-cylinders are connected in a uniform manner (see the purple arrows), the dynamic on the first tower consists in going from the bottom to the left. For an associated odomutant, we divide this tower in q_1 subtowers $[\bullet,0]_2,\dots,[\bullet,q_1-1]_2$ according to the value of the second coordinate, and the dynamic in each of these subtowers does not necessarily consist in going from the bottom to the top. In the subtower $[\bullet,1]_2$ of this example, the odomutant maps $[0,1]_2$ to $[2,1]_2$, which is then mapped to $[1,1]_2$, which is finally mapped to $[3,1]_2$.
At the second step, the odometer connects the subtowers in the following way (see the orange arrows): the end of the subtower associated to $[\bullet,i]_2$ is mapped to the begining of the subtower associated to $[\bullet,i+1]_2$. An odomutant divides these subtowers so that it has many possibilities to connect them.

$\{0,1,\dots,q_n-1\}$. We consider the odometer $S\colon X \to X$ on this space. Recall that it is defined by

$$Sx = \left\{ \begin{array}{ll} (0,\dots,0,x_i+1,x_{i+1},\dots) & \text{if } i \coloneqq \min\{j \geqslant 0 \mid x_j \neq q_j - 1\} \text{ is finite} \\ (0,0,0,\dots) & \text{if } x = (q_0-1, q_1-1, q_2-1,\dots) \end{array} \right. ,$$

and it is a μ-preserving homeomorphism.

In this section, we introduce new systems that we call odomutants, defined from S with successive distortions of its orbits, encoded by the following maps ψ and ψ_n (for $n \geqslant 0$).

For every $n \geqslant 0$, we fix a sequence $\left(\sigma_i^{(n)}\right)_{0\leqslant i\leqslant q_{n+1}-1}$ of permutations of the set $\{0,1,\dots,q_n-1\}$, and we introduce

$$\psi_n\colon \begin{cases} X & \to X \\ x=(x_0,x_1,\dots) & \mapsto (\sigma^{(0)}_{x_1}(x_0),\sigma^{(1)}_{x_2}(x_1),\sigma^{(2)}_{x_3}(x_2),\dots,\sigma^{(n)}_{x_{n+1}}(x_n),x_{n+1},x_{n+2},\dots) \end{cases}.$$

It is not difficult to see that ψ_n is a homeomorphism and preserves the measure μ, its inverse is given by

$$\psi_n^{-1}\colon \begin{cases} X & \to X \\ x=(x_0,x_1,\dots) & \mapsto (z_0(x),z_1(x),\dots,z_n(x),x_{n+1},x_{n+2},\dots) \end{cases}$$

with $z_i(x)$ inductively defined by

$$\begin{aligned} z_n(x) &\coloneqq \left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(x_n), \\ z_i(x) &\coloneqq \left(\sigma^{(i)}_{z_{i+1}(x)}\right)^{-1}(x_i) \text{ for every } i\in\{0,1,\dots,n-1\}. \end{aligned} \tag{1}$$

The following computations motivate the definition of odomutants. Let us respectively set the **minimal** and **maximal** points of X:

$$x^- \coloneqq (0,0,0,\dots) \text{ and } x^+ \coloneqq (q_0-1,q_1-1,q_2-1,\dots).$$

We define the following sets

$$\begin{aligned} X_n^- &\coloneqq \{x\in X \mid (x_0,\dots,x_n)\neq(x_0^-,\dots,x_n^-)\}, \\ X_n^+ &\coloneqq \{x\in X \mid (x_0,\dots,x_n)\neq(x_0^+,\dots,x_n^+)\}, \\ X_\infty^- &\coloneqq X\backslash\{x^-\} \text{ and } X_\infty^+ \coloneqq X\backslash\{x^+\}. \end{aligned}$$

It is not difficult to see that X_∞^+ is the increasing union of the sets X_n^+, so for every $x\in X_\infty^+$, we denote by $N^+(x)$ the least integer $n\geqslant 0$ satisfying $x\in X_n^+$. This also holds for X_∞^- and X_n^-, and $N^-(x)$ is defined similarly.

Let $x\in\psi^{-1}(X_\infty^+)$ and $N\coloneqq N^+(\psi(x))$. By definition of N, for every $n\geqslant N$, $S\psi_n(x)$ is equal to

$$(\underbrace{\mathbf{0,\dots,0}}_{N\text{ times}},\ \sigma^N_{x_{N+1}}(x_N)\mathbf{+1},\ \sigma^{(N+1)}_{x_{N+2}}(x_{N+1}),\ \dots,\ \sigma^{(n)}_{x_{n+1}}(x_n),\ x_{n+1},\ x_{n+2},\ \dots).$$

Using (1), we get

$$\psi_n^{-1}S\psi_n(x) = (y_0^{(n)}(x),\dots,y_n^{(n)}(x),x_{n+1},x_{n+2},\dots)$$

with $y_i^{(n)}(x)$ inductively defined by

$$\begin{aligned} & y_n^{(n)}(x) \coloneqq \left(\sigma^{(n)}_{x_{n+1}}\right)^{-1}(\sigma^{(n)}_{x_{n+1}}(x_n)), \\ \forall\ N+1\leqslant i\leqslant n-1,\quad & y_i^{(n)}(x) \coloneqq \left(\sigma^{(i)}_{y^{(n)}_{i+1}(x)}\right)^{-1}(\sigma^{(i)}_{x_{i+1}}(x_i)), \\ & y_N^{(n)}(x) \coloneqq \left(\sigma^{(N)}_{y^{(n)}_{N+1}(x)}\right)^{-1}(\sigma^{(N)}_{x_{N+1}}(x_N)+1), \\ \forall\ 0\leqslant i\leqslant N-1,\quad & y_i^{(n)}(x) \coloneqq \left(\sigma^{(i)}_{y^{(n)}_{i+1}(x)}\right)^{-1}(0). \end{aligned}$$

By induction, it is easy to get $(y^{(n)}_{N+1}(x),\dots,y^{(n)}_n(x)) = (x_{N+1},\dots,x_n)$ and this implies the following simplification: $\psi_n^{-1}S\psi_n(x)$ is equal to $(y_0^{(n)}(x),\dots,y_N^{(n)}(x),x_{N+1},x_{N+2},\dots)$ with $y_i^{(n)}(x)$ inductively defined by

$$\begin{aligned} & y_N^{(n)}(x) \coloneqq \left(\sigma^{(N)}_{x_{N+1}}\right)^{-1}(\sigma^{(N)}_{x_{N+1}}(x_N)+1), \\ \forall\ 0\leqslant i\leqslant N-1,\quad & y_i^{(n)}(x) \coloneqq \left(\sigma^{(i)}_{y^{(n)}_{i+1}(x)}\right)^{-1}(0). \end{aligned}$$

Finally, $(y_0^{(n)}(x),\dots,y_N^{(n)}(x))$ does not depend on the integer $n\geqslant N^+(\psi(x))$.

Definition 4.1. For every $x\in\psi^{-1}(X_\infty^+)$, let us define

$$Tx \coloneqq \psi_n^{-1}S\psi_n(x)$$

for any $n\geqslant N^+(\psi(x))$. The map T is called the **odomutant** associated to the odometer S and the sequences of permutations $\left(\sigma_i^{(n)}\right)_{0\leqslant i\leqslant q_{n+1}-1}$ for $n\geqslant 0$.

4.2 Odomutants as probability measure-preserving bijection

Let us introduce the map

$$\psi\colon \left\{ \begin{array}{lcl} X & \to & X \\ x=(x_0,x_1,\dots) & \mapsto & \left(\sigma^{(n)}_{x_{n+1}}(x_n)\right)_{n\geqslant 0} \end{array} \right. ,$$

namely $\psi(x) = \lim_{n\to+\infty} \psi_n(x)$ for every $x \in X$. The map ψ is continuous but is not invertible in full generality. The map ψ also have the following properties.

Proposition 4.2. *$\psi\colon X \to X$ preserves the probability measure μ and is onto.*

Proof of Proposition 4.2. To prove that μ is ψ-invariant, it suffices to prove the equality $\mu(\psi^{-1}(A)) = \mu(A)$ when A is a cylinder. If A is an $(n+1)$-cylinder, then $\psi^{-1}(A) = \psi_n^{-1}(A)$, so the ψ-invariance follows from the ψ_n-invariance for all $n \geqslant 0$.

Given $y \in X$, let us find $x \in X$ such that $\psi(x) = y$. By definition, for every $n \geqslant 0$, $\psi(\psi_n^{-1}(y))$ is in the cylinder $[y_0,\dots,y_n]_{n+1}$, so $\psi(\psi_n^{-1}(y)) \underset{n\to+\infty}{\to} y$. By compactness, there exists a convergent subsequence of $\left(\psi_n^{-1}(y)\right)_{n\geqslant 0}$, of limit $x \in X$, and we have $\psi(x) = y$ since ψ is continuous. □

Proposition 4.3. *T is a bijection from $\psi^{-1}(X_\infty^+)$ to $\psi^{-1}(X_\infty^-)$, its inverse is given by*

$$T^{-1}y = \psi_n^{-1}S^{-1}\psi_n(y)$$

for every $y \in \psi^{-1}(X_\infty^-)$ and any $n \geqslant N^-(\psi(y))$. Moreover T is an element of $\mathrm{Aut}(X,\mu)$ and ψ is a factor map from T to S.

Proof of Proposition 4.3. The equality $\psi_n(Tx) = S\psi_n(x)$ implies $\psi(Tx) = S\psi(x)$ since ψ_n converges pointwise to ψ. Moreover, the map ψ preserves the measure μ and is onto (see Proposition 4.2). Thus, assuming that T is in $\mathrm{Aut}(X,\mu)$, S is a factor of T via the factor map ψ.

T is injective on $\psi^{-1}(X_\infty^+)$. Indeed, X_∞^+ is the increasing union of the sets X_n^+, and for every $n \geqslant 0$, T and $\psi_n^{-1}S\psi_n$ coincide on X_n^+, so the injectivity of T on $\psi^{-1}(X_\infty^+)$ follows from the injectivity of S and the maps ψ_n and ψ_n^{-1}.

For $x \in \psi^{-1}(X_\infty^+)$, we have $\psi(Tx) = S\psi(x)$ and $\psi(x) \neq x^+$, so $\psi(Tx)$ is not equal to x^-. Conversely, for $y \in \psi^{-1}(X_\infty^-)$, the element $x \coloneqq \psi_n^{-1}S^{-1}\psi_n(y)$ does not depend on the choice of an integer $n \geqslant N^-(\psi(y))$ (these are the same computations as before Definition 4.1) and satisfies $Tx = y$.

By ψ-invariance, the sets $\psi^{-1}(X_\infty^+)$ and $\psi^{-1}(X_\infty^-)$ have full measure, so $T\colon X \to X$ is a bijection up to measure zero. It follows again from the properties of S and the maps ψ_n that T is bimeasurable and preserves the measure μ. □

Since every odometer is coalescent (see Theorem 3.4), we get the following:

Corollary 4.4. *Let T be an odomutant built from the odometer S on $X = \prod_{n\geqslant 0}\{0,\dots,q_n-1\}$ and families of permutations $\left(\sigma_i^{(n)}\right)_{0\leqslant i\leqslant q_{n+1}}$, for $n \geqslant 0$. The following assertions are equivalent:*

1. *T is conjugate to S;*
2. *$\psi\colon x \in X \mapsto \left(\sigma^{(n)}_{x_{n+1}}(x_n)\right)_{n\geqslant 0} \in X$ is an isomorphism;*
3. *$\psi\colon x \in X \mapsto \left(\sigma^{(n)}_{x_{n+1}}(x_n)\right)_{n\geqslant 0} \in X$ is injective almost everywhere.*

Question 4.5. Is it possible to find a necessary and sufficient condition on the permutations $\sigma_i^{(n)}$ (for $n \geqslant 0$ and $0 \leqslant i \leqslant q_{n+1}-1$) for the factor map ψ to be an isomorphism?

In the proof of Theorem B, for the odomutant to extend to an homeomorphism, the permutations will have common fixed point: for every $n \geqslant 0$ and every $0 \leqslant i \leqslant q_{n+1}$, we assume $\sigma_i^{(n)}(0) = 0$ and $\sigma_i^{(n)}(q_n - 1) = q_n - 1$. In this context, we can find a sufficient condition for ψ to be an isomorphism:

Lemma 4.6. *For every $n \geqslant 0$, we set*

$$F_n \coloneqq \{x_n \in \{0, \ldots, q_n - 1\} \mid \forall x_{n+1} \in \{0, \ldots, q_{n+1} - 1\},\ \sigma_{x_{n+1}}^{(n)}(x_n) = x_n\}.$$

If the series $\sum \frac{|F_n|}{q_n}$ diverges, then ψ is an isomorphism between S and T.

So, when $|F_n| = 2$ (as in the proof of Theorem B), in order to get an odomutant of positive entropy, we need the sequence $(q_n)_{n\geqslant 0}$ to increase quickly enough, otherwise we get an odomutant T conjugate to S.

Proof of Lemma 4.6. By the Borel-Cantelli lemma, the set

$$X_0 \coloneqq \{(x_n)_{n\geqslant 0} \in X \mid x_n \in F_n \text{ for infinitely many integers } n\}$$

has full measure. It is also S-, T- and ψ-invariant and it is easy to check that $\psi\colon X_0 \to X_0$ is a bijection, using the fact that the equality $\sigma_{x_{n+1}}^{(n)}(x_n) = y_n$ implies $x_n = y_n$ when y_n is in F_n. ☐

However the condition in Lemma 4.6 is not a necessary condition. Counter example: whatever the sequence (q_n) can be, if for every $n \geqslant 0$, $\sigma_i^{(n)}$ does no depend on $0 \leqslant i \leqslant q_{n+1}$, then ψ is an isomorphism.

4.3 Point spectrum

Theorem 4.7. *Let T be an odomutant built from the odometer S on $X = \prod_{n\geqslant 0}\{0, \ldots, q_n - 1\}$. Then T and S have the same point spectrum.*

We do not give the proof and refer the reader to the upcoming article about odomutants.

Using Halmos-von Neumann Theorem, we get the following corollary.

Corollary 4.8. *Let T be an odomutant built from the odometer S on $X = \prod_{n\geqslant 0}\{0, \ldots, q_n - 1\}$. The following assertions are equivalent:*

1. *T is conjugate to an odometer;*
2. *T is conjugate to S.*

4.4 Orbit equivalence between odometers and odomutants

In this section, we prove that an odomutant and its associated odometer have the same orbits. Moreover, given a non-decreasing map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, we give sufficient conditions for the cocycles to be φ-integrable.

☠ Given the frightening formulas in Proposition 4.9 and in Condition (C1) of Proposition 4.10, the reader may first refer to Figure 5 for a schematic view of one of the cocycles.

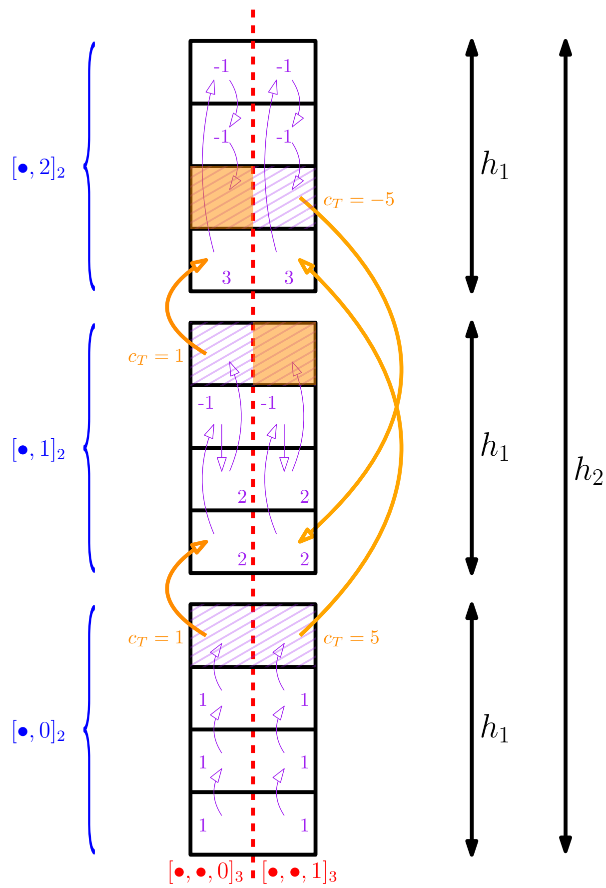


Figure 5: Let us consider the example in Figure 4. The length of the arrows are exactly the value of the cocycle c_T: the numbers in purple (resp. orange) are the values of the cocycle given by the purple (resp. orange) arrows. At the first step, the odomutant is defined on a subset of X (so of measure $\leqslant 1$) where $|c_T| \leqslant h_1$. At the second step, we define the odomutant on new points lying in a subset of the purple area (of measure $\leqslant 1/h_1$) where $|c_T| \leqslant h_2$, and so on. Hence Condition (C2) that we will use for Theorems B and C.
Notice that if q_1 is very large (so there are many subtowers representing $[\bullet, i]_2$ for $0 \leqslant i \leqslant q_1 - 1$) and if an orange arrow connects two consecutive towers, then the bound $|c_T| \leqslant h_2$ is too coarse. This is the reason why we have the finer Condition (C1) which will enable us to exploit the sublinearity of the map φ in Theorem A.

Proposition 4.9. *For all $x \in \psi^{-1}(X_\infty^+)$, we have $Tx = S^{c_T(x)}x$ where the integer $c_T(x)$ is defined by*

$$c_T(x) = \sum_{i=0}^{N_1} h_i(y_i(x) - x_i) \tag{2}$$

with $N_1 \coloneqq N^+(\psi(x))$ and $y_0, \dots, y_{N_1}(x)$ inductively defined by

$$y_{N_1}(x) \coloneqq \left(\sigma^{(N_1)}_{x_{N_1+1}}\right)^{-1}(\sigma^{(N_1)}_{x_{N_1+1}}(x_{N_1}) + 1),$$
$$\forall\ 0 \leqslant i \leqslant N_1 - 1, \quad y_i(x) \coloneqq \left(\sigma^{(i)}_{y_{i+1}(x)}\right)^{-1}(0).$$

For all $x \in X_\infty^+$, let us define the integer $c_S(x)$ by:

$$\begin{aligned} c_S(x) = \quad & h_{N_2}\left(\sigma^{(N_2)}_{x_{N_2+1}}(1 + x_{N_2}) - \sigma^{(N_2)}_{x_{N_2+1}}(x_{N_2})\right) + h_{N_2-1}\left(\sigma^{(N_2-1)}_{1+x_{N_2}}(0) - \sigma^{(N_2-1)}_{x_{N_2}}(x_{N_2-1})\right) \\ & + \sum_{i=0}^{N_2-2} h_i\left(\sigma^{(i)}_0(0) - \sigma^{(i)}_{x_{i+1}}(x_i)\right) \end{aligned} \tag{3}$$

with $N_2 \coloneqq N^+(x)$. Then we have $Sx = T^{c_S(x)}x$ for every $x \in X_\infty^+$.

Proof of Proposition 4.9. For $x \in \psi^{-1}(X_\infty^+)$, the value of $c_T(x)$ follows from the computations before Definition 4.1. For $x \in X_\infty^+$ and $N_2 \coloneqq N^+(x)$, we have

$$x = (q_0 - 1, \dots, q_{N_2-1} - 1, \underbrace{x_{N_2}}_{\neq q_{N_2}-1}, x_{N_2+1}, x_{N_2+2}, \dots) \text{ and } Sx = (0, \dots, 0, 1 + x_{N_2}, x_{N_2+1}, x_{N_2+2}, \dots),$$

so for every $n \geqslant N_2$:

$$\psi_n(x) = (\sigma_{x_1}^{(0)}(x_0), \dots, \sigma_{x_{N_2}}^{(N_2-1)}(x_{N_2-1}), \sigma_{x_{N_2+1}}^{(N_2-1)}(x_{N_2}), \dots)$$

$$\text{and } \psi_n(Sx) = (\sigma_0^{(0)}(0), \dots, \sigma_0^{(N_2-2)}(0), \sigma_{1+x_{N_2}}^{(N_2-1)}(0), \sigma_{x_{N_2+1}}^{(N_2-1)}(1 + x_{N_2}), \dots)$$

and it is straightforward to get $\varphi_n(Sx) = S^{c_S(x)}\varphi_n(x)$, thus implying $Sx = T^{c_S(x)}x$. □

Theorem 4.10. *The map $\Psi \coloneqq id_X$ is an orbit equivalence between T and S. Moreover, given an non-decreasing map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, this orbit equivalence is φ-integrable if one of the following two conditions is satisfied:*

(C1) the series

$$\sum_{n \geqslant 0} \frac{1}{h_{n+2}} \sum_{\substack{0 \leqslant x_n \leqslant q_n - 1, \\ 0 \leqslant x_{n+1} \leqslant q_{n+1} - 1, \\ \sigma_{x_{n+1}}^{(n)}(x_n) \neq q_n - 1}} \varphi\left(h_n \left(1 + \left|\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1} (\sigma_{x_{n+1}}^{(n)}(x_n) + 1) - x_n\right|\right)\right)$$

$$\text{and } \sum_{n \geqslant 0} \frac{1}{h_{n+2}} \sum_{\substack{0 \leqslant x_n \leqslant q_n - 2, \\ 0 \leqslant x_{n+1} \leqslant q_{n+1} - 1}} \varphi\left(h_n \left(1 + \left|\sigma_{x_{n+1}}^{(n)}(1 + x_n) - \sigma_{x_{n+1}}^{(n)}(x_n)\right|\right)\right)$$

converge;

(C2) the series $\sum \frac{\varphi(h_{n+1})}{h_n}$ converges.

Proof of Theorem 4.10. By Proposition 4.9, the set of points $x \in X$ satisfying $Tx = S^{c_T(x)}x$ and $Sx = T^{c_S(x)}x$ for integers $c_T(x)$ and $c_S(x)$ defined by (2) and (3) have full measure, so the map id_X is an orbit equivalence between S and T.

The value of $c_T(x)$ gives the following bound:

$$|c_T(x)| \leqslant h_{N_1} \left|\left(\sigma_{x_{N_1+1}}^{(N_1)}\right)^{-1} (\sigma_{x_{N_1+1}}^{(N_1)}(x_{N_1}) + 1) - x_{N_1}\right| + \underbrace{\sum_{i=0}^{N_1-1} h_i \, |y_i(x) - x_i|}_{\leqslant h_{N_1}} \tag{4}$$

with $N_1 = N^+(\psi(x))$. Given $n \geqslant 0$, $z_n \in \{0, \dots, q_n - 1\}$ and $z_{n+1} \in \{0, \dots, q_{n+1} - 1\}$, we have

$$\mu(\{x \in X \mid N^+(\psi(x)) = n, x_n = z_n, x_{n+1} = z_{n+1}\}) \leqslant \frac{1}{h_{n+2}}.$$

We finally get

$$\begin{aligned}
&\int_X \varphi(|c_T(x)|)\mathrm{d}\mu(x) \\
= \ & \sum_{n \geqslant 0} \sum_{\substack{0 \leqslant z_n \leqslant q_n - 1, \\ 0 \leqslant z_{n+1} \leqslant q_{n+1} - 1, \\ \sigma_{z_{n+1}}^{(n)}(z_n) \neq q_n - 1}} \int_{\substack{N^+(\psi(x)) = n, \\ x_n = z_n, \\ x_{n+1} = z_{n+1}}} \varphi(|c_T(x)|)\mathrm{d}\mu(x) \\
\leqslant \ & \sum_{n \geqslant 0} \frac{1}{h_{n+2}} \sum_{\substack{0 \leqslant z_n \leqslant q_n - 1, \\ 0 \leqslant z_{n+1} \leqslant q_{n+1} - 1, \\ \sigma_{z_{n+1}}^{(n)}(z_n) \neq q_n - 1}} \varphi\left(h_n \left(1 + \left|\left(\sigma_{z_{n+1}}^{(n)}\right)^{-1} (\sigma_{z_{n+1}}^{(n)}(z_n) + 1) - z_n\right|\right)\right).
\end{aligned}$$

From Inequality (4), we also get $|c_T(x)| \leqslant h_{N_1+1}$ and the following coarser bound:

$$\begin{aligned}
&\int_X \varphi(|c_T(x)|)\mathrm{d}\mu(x) \\
= &\sum_{n\geqslant 0} \sum_{\substack{0\leqslant z_n \leqslant q_n-1,\\ 0\leqslant z_{n+1}\leqslant q_{n+1}-1,\\ \sigma^{(n)}_{z_{n+1}}(z_n)\neq q_n-1}} \int_{\substack{N^+(\psi(x))=n,\\ x_n=z_n,\\ x_{n+1}=z_{n+1}}} \varphi(|c_T(x)|)\mathrm{d}\mu(x) \\
\leqslant &\sum_{n\geqslant 0} \frac{1}{h_{n+2}} \sum_{\substack{0\leqslant z_n \leqslant q_n-1,\\ 0\leqslant z_{n+1}\leqslant q_{n+1}-1,\\ \sigma^{(n)}_{z_{n+1}}(z_n)\neq q_n-1}} \varphi\left(h_{n+1}\right) \\
\leqslant &\sum_{n\geqslant 0} \frac{1}{h_n}\varphi(h_{n+1}).
\end{aligned}$$

For the other cocycle, we have

$$\begin{aligned}
|c_S(x)| \leqslant\ & h_{N_2}\left|\sigma^{(N_2)}_{x_{N_2+1}}(1+x_{N_2}) - \sigma^{(N_2)}_{x_{N_2+1}}(x_{N_2})\right| \\
&+ \underbrace{h_{N_2-1}\left|\sigma^{(N_2-1)}_{1+x_{N_2}}(0) - \sigma^{(N_2-1)}_{x_{N_2}}(x_{N_2-1})\right| + \sum_{i=0}^{N_2-2} h_i \left|\sigma^{(i)}_0(0) - \sigma^{(i)}_{x_{i+1}}(x_i)\right|}_{\leqslant h_{N_2}}. \qquad (5)
\end{aligned}$$

with $N_2 = N^+(x)$. Moreover it is easy to get

$$\mu(\{x \in X \mid N^+(x) = n, x_n = z_n, x_{n+1} = z_{n+1}\}) \leqslant \frac{1}{h_{n+2}}$$

for every $n \geqslant 0$, $z_n \in \{0, \dots, q_n - 1\}$ and $z_{n+1} \in \{0, \dots, q_{n+1} - 1\}$. Thus we find a bound on the φ-integral of c_S with the same method as c_T. □

5 Orbit equivalence with almost integrable orbit equivalence cocycles

In this section, we prove that being orbit equivalent to an odometer, with almost integrable cocycles, does not imply being flip-conjugate to it.

Theorem 5.1 (C.). *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map and S an odometer. There exists a probability measure-preserving transformation T such that S and T are φ-integrably orbit equivalent but not flip-conjugate.*

For Theorems C and B, some invariants (loose Bernoullicity property, entropy) ensure that we build an odomutant T which is not flip-conjugate to the associated odometer S. For Theorem 5.1, we use the following facts

1. $\psi\colon x \in X \to (\sigma^{(0)}_{x_1}(x_0), \sigma^{(1)}_{x_2}(x_1), \sigma^{(2)}_{x_3}(x_2), \dots) \in X$ is a factor map from an odomutant T to its associated odometer S;
2. every odometer is coalescent (see Theorem 3.4).

The goal is to find families of permutations $\left(\sigma^{(n)}_{x_{n+1}}\right)_{0\leqslant x_{n+1}\leqslant q_{n+1}-1}$, for $n \geqslant 0$, such that the factor map ψ is not an isomorphism, with φ-integrable cocycles for the orbit equivalence between S and T.

Lemma 5.2. *Let $(q_n)_{n\geqslant 0}$ be a sequence of integers greater or equal to 2. For every $n \geqslant 0$, let $\left(\sigma^{(n)}_{x_{n+1}}\right)_{0\leqslant x_{n+1}\leqslant q_{n+1}-1}$ be a family of permutations of the set $\{0,1,\ldots,q_n-1\}$, defined by:*

$$\forall x_{n+1} \in \{0,\ldots,q_{n+1}-1\},\ \forall i \in \{0,\ldots,q_n-1\},\ \sigma^{(n)}_{x_{n+1}}(i) = i + x_{n+1} \bmod q_n.$$

Assume that the infinite product $\prod_{n\geqslant 0}\left(1-\frac{1}{q_n}\right)$ converges[a]*. Then $\psi\colon x \in X \to (\sigma^{(n)}_{x_{n+1}}(x_n))_{n\geqslant 0} \in X$ is not injective almost everywhere.*

[a] By definition, the infinite product $\prod_{n\geqslant 0}\left(1-\frac{1}{q_n}\right)$ converges if the sequence $\left(\prod_{k=0}^{n}\left(1-\frac{1}{q_k}\right)\right)_{n\geqslant 0}$ converges to a nonzero real number.

Proof of Lemma 5.2. **Idea:** Find a partial isomorphism θ which maps every point x of its domain to another point of $\psi^{-1}(\psi(x))$.

Let $Y_1 \coloneqq \{x \in X \mid \forall n \geqslant 0, x_n \neq (q_n-1)\mathbb{1}_{n \text{ is even}}\}$ and $Y_2 \coloneqq \{x \in X \mid \forall n \geqslant 0, x_n \neq (q_n-1)\mathbb{1}_{n \text{ is odd}}\}$. It is straightforward to check that

$$\mu(Y_1) = \mu(Y_2) = \prod_{n\geqslant 0}\left(1-\frac{1}{q_n}\right) > 0.$$

Let $\theta\colon X \to X$ defined by:

$$\theta(x) \coloneqq (x_n + (-1)^n \bmod q_n)_{n\geqslant 0}.$$

The map θ is in $\mathrm{Aut}(X,\mu)$ since X can be seen as the compact group $\prod_{n\geqslant 0}\mathbb{Z}/q_n\mathbb{Z}$, with its Haar probability measure μ and θ as the translation by $((-1)^n)_{n\geqslant 0}$. Moreover, θ is a bijection from Y_1 to Y_2 and we have $\psi(\theta(x)) = \psi(x)$ for all $x \in Y_1$.

Let us prove by contradiction that ψ is not injective almost everywhere. Assume that ψ is injective on a measurable set X_0 of full measure. This hypothesis and the equality $\psi\circ\theta = \psi$ on Y_1 imply that the sets X_0 and $\theta(X_0 \cap Y_1)$ are disjoint. This finally gives

$$\mu((X_0)^c) \geqslant \mu\left(\theta(X_0\cap Y_1)\right) = \mu(X_0 \cap Y_1) = \mu(Y_1) > 0$$

and we get a contradiction since $(X_0)^c$ has zero measure. ☐

Before the proof of Theorem A, we use a lemma stated in [CJLMT23] and which enables us to reduce to the case where the sublinear map φ is non-decreasing (actually the statement is stronger but we only need the monotonicity).

Lemma 5.3 (Lemma 2.12 in [CJLMT23])**.** *Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Then there is a sublinear non-decreasing function $\tilde{\varphi}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \leqslant \tilde{\varphi}(t)$ for all t large enough.*

Proof of Theorem A. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear map. If $\tilde{\varphi}$ is another sublinear map satisfying $\varphi(t) = O(\tilde{\varphi(t)})$, then $\tilde{\varphi}$-integrability implies φ-integrability. Therefore, by Lemma 5.3, we assume without loss of generality that φ is non-decreasing.

Let $(q_n)_{n\geqslant 0}$ be a sequence of integers greater or equal to 2 and S the odometer on $X \coloneqq \prod_{n\geqslant 0}\{0,1,\ldots,q_n-1\}$. Halmos-von Neumann Theorem implies that S is conjugate to the odometer on $\prod_{n\geqslant 0}\{0,1,\ldots,q_{i_{n-1}}\cdots q_{i_n-1}-1\}$ for any increasing sequence $(i_n)_{n\geqslant 0}$ satisfying $i_0 = 0$. Therefore, we can assume without loss of generality that the integers q_n are sufficiently large so that they satisfy the following properties:

1. $\prod_{n\geqslant 0}\left(1-\frac{1}{q_n}\right)$ converges[a];
2. the series $\sum \dfrac{\varphi(2h_n)}{h_n}$ converges.

Let T be the odomutant built from S and the same families $\left(\sigma_{x_{n+1}}^{(n)}\right)_{0\leqslant x_{n+1}\leqslant q_{n+1}-1}$ as in Lemma 5.2. By this lemma and Theorem 3.4, S and T are not conjugate. Since S is conjugate to its inverse S^{-1} (by Halmos-von Neumann Theorem), S and T are not flip-conjugate.

It remains to quantify the cocycles, using Condition (C1) of Theorem 4.10. Let $n \geqslant 0$ and $x_{n+1} \in \{0,\dots,q_{n+1}-1\}$, and $i \in \{0,\dots,q_n-1\}$ such that $x_{n+1} = i \bmod q_n$. For every $x \in \{0,\dots,q_n-2\}\backslash\{q_n-i-1\}$, we have

$$\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1}(\sigma_{x_{n+1}}^{(n)}(x_n)+1)-x_n = \sigma_{x_{n+1}}^{(n)}(1+x_n)-\sigma_{x_{n+1}}^{(n)}(x_n) = 1.$$

For $x_n = q_n-1$, we consider the following bounds:

$$\left|\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1}(\sigma_{x_{n+1}}^{(n)}(x_n)+1)-x_n\right| \leqslant q_n$$

$$\text{and } \left|\sigma_{x_{n+1}}^{(n)}(1+x_n)-\sigma_{x_{n+1}}^{(n)}(x_n)\right| \leqslant q_n.$$

We finally get

$$\begin{aligned}
&\sum_{n\geqslant 0}\frac{1}{h_{n+2}}\sum_{\substack{0\leqslant x_n\leqslant q_n-1,\\ 0\leqslant x_{n+1}\leqslant q_{n+1}-1,\\ \sigma_{x_{n+1}}^{(n)}(x_n)\neq q_n-1}}\varphi\left(h_n\left(1+\left|\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1}(\sigma_{x_{n+1}}^{(n)}(x_n)+1)-x_n\right|\right)\right)\\
=\;&\sum_{n\geqslant 0}\frac{1}{h_{n+2}}\sum_{\substack{0\leqslant x_n\leqslant q_n-2,\\ 0\leqslant x_{n+1}\leqslant q_{n+1}-1\\ x_n\neq q_n-i-1}}\varphi\left(h_n\left(1+\left|\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1}(\sigma_{x_{n+1}}^{(n)}(x_n)+1)-x_n\right|\right)\right)\\
\leqslant\;&\sum_{n\geqslant 0}\frac{1}{h_{n+2}}\sum_{0\leqslant x_{n+1}\leqslant q_{n+1}-1}((q_n-2)\varphi(2h_n)+\varphi(h_n(1+q_n)))\\
\leqslant\;&\sum_{n\geqslant 0}\frac{\varphi(2h_n)}{h_n}+\sum_{n\geqslant 0}\frac{\varphi(2h_{n+1})}{h_{n+1}} < +\infty
\end{aligned}$$

and similarly

$$\sum_{n\geqslant 0}\frac{1}{h_{n+2}}\sum_{\substack{0\leqslant x_n\leqslant q_n-2,\\ 0\leqslant x_{n+1}\leqslant q_{n+1}-1}}\varphi\left(h_n\left(1+\left|\sigma_{x_{n+1}}^{(n)}(1+x_n)-\sigma_{x_{n+1}}^{(n)}(x_n)\right|\right)\right) < \infty,$$

so S and T are φ-integrably orbit equivalent. □

6 On non-preservation of entropy under almost log-integrable orbit equivalence

We want to prove Theorem B:

Theorem 6.1 (C.). *Let S be the universal odometer. Then there exists a Cantor minimal homeomorphism T such that*

1. $\mathrm{h_{top}}(T) > 0$;
2. *there exists a strong orbit equivalence between S and T, which is φ_m-integrable for all integers $m \geqslant 0$, where φ_m denotes the map $t \to \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log\circ\ldots\circ\log$ (m times).*

This is a weaker version of a more general theorem (see Theorem 2.16).

6.1 Strong orbit equivalence

Proposition 6.2. *Assume that two aperiodic measurable bijections S and T on a Borel space X are orbit equivalent in the following stronger way: S and T are defined on the whole X and the equality $\mathrm{Orb}_S(x) = \mathrm{Orb}_T(x)$ holds for every $x \in X$.*[a] *Then S is uniquely ergodic if and only if T is uniquely ergodic. In this case, S and T have the same invariant probability measure.*

[a] This is stronger than asking this property up to a null set.

Proof of Proposition 6.2. Assume that S is uniquely ergodic and denote by μ its only invariant probability measure. The cocycle $c_S\colon X \to \mathbb{Z}$ is defined on the whole X and is measurable. Let ν be a T-invariant probability measure. For every measurable set A, we have

$$\begin{aligned} \nu(S(A)) &= \sum_{k\in\mathbb{Z}} \nu(S(A \cap \{c_S = k\})) \\ &= \sum_{k\in\mathbb{Z}} \nu(T^k(A \cap \{c_S = k\})) \\ &= \sum_{k\in\mathbb{Z}} \nu(A \cap \{c_S = k\}) \\ &= \nu(A), \end{aligned}$$

so ν is S-invariant and is equal to μ. Therefore T is uniquely ergodic and μ is its only invariant probability measure. □

For instance, strong orbit equivalence is a form of orbit equivalence, introduced in a topological framework by Giordano, Putnam and Skau [GPS95], to which Proposition 6.2 applies. The definition is the following.

Definition 6.3. Two Cantor minimal homeomorphisms (X, S) and (Y, T) are strongly orbit equivalent if there exists a homeomorphism $\Psi\colon X \to Y$ such that S and $\Psi^{-1}T\Psi$ have the same orbits on X and the associated cocycles each have at most one point of discontinuity.

Boyle proved in his thesis [Boy83] that strong orbit equivalence with continuous cocycles boils down to topological flip-conjugacy, namely S is topologically conjugate to T or to T^{-1}. As mentioned in the introduction, the classification up to strong orbit equivalence is fully understood, with complete invariants such as the dimension group (see [GPS95]).

6.2 Extension to a homeomorphism on the Cantor set

Proposition 6.4. *Assume that $\sigma_i^{(n)}(0) = 0$ and $\sigma_i^{(n)}(q_n - 1) = q_n - 1$ for every $n \geqslant 0$ and every $0 \leqslant i \leqslant q_n - 1$. Then the odomutant T admits a unique extension which is a homeomorphism on the whole compact set $X = \prod_{n\geqslant 0} \{0, 1, \ldots, q_n - 1\}$. It is furthermore strongly orbit equivalent to the associated odometer S. In particular, it follows from Proposition 6.2 that T is uniquely ergodic.*

Remark 6.5. In this case, the equality $S \circ \psi(x) = \psi \circ T(x)$ holds for all $x \in X$.

Proof of Proposition 6.4. Since, for every $n \geqslant 0$, the points 0 and $q_n - 1$ are fixed by the n-th permutations, x^- is the only point $x \in X$ satisfying $\psi(x) = x^-$ and x^+ is the only point $x \in X$ satisfying $\psi(x) = x^+$. This implies that we have

$$\psi^{-1}(X_\infty^-) = X_\infty^- = X \backslash \{x^-\} \text{ and } \psi^{-1}(X_\infty^+) = X_\infty^+ = X \backslash \{x^+\},$$

and T is a bijection from $X\backslash\{x^+\}$ to $X\backslash\{x^-\}$, so we set $Tx^+ \coloneqq x^-$. The map $T\colon X \to X$ is now a well-defined bijection.

It is not difficult to prove that T is a homeomorphism.

By Proposition 4.9, we have $Tx = S^{c_T(x)}x$ and $Sx = T^{c_S(x)}$ for every $x \in X_\infty^+$, with $c_T(x)$ and $c_S(x)$ defined by (2) and (3). These relations are extended at x^+, with $c_T(x) = c_S(x) = 1$. Thus S and T have the same orbits and it is clear that the cocycles are continuous on X_∞^+ (x^+ is the only point of discontinuity).[2] □

6.3 Coding map with respect to the partition in 1-cylinders

Topological entropy is often easier to compute than measure-theoretical entropy. In this context, the space $X = \prod_{n\geqslant 0}\{0,1,\dots q_n-1\}$ is a Cantor set, so it admits open covers which are partitions: the partition in n-cylinders (for every $n \geqslant 0$). As explained in Example A.19, when the open cover $\mathcal{P}$ that we consider is a partition, then the minimal cardinal of an open subcover is actually the cardinal of this partition, so the entropy of a topological system T with respect to $\mathcal{P}$ exactly consists in studying the asymptotical behaviour of the cardinality of

$$\mathcal{P}_0^{n-1} \coloneqq \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{P})$$

Moreover, the cardinality of $\mathcal{P}_0^{n-1}$ is the number of words $(P_0,\dots,P_{n-1})$ such that there exists $x \in X$ satisfying $x \in P_0, Tx \in P_1, \dots, T^{n-1}x \in P_{n-1}$, namely the cardinality of $\{[\mathcal{P}]_n(x) \mid x \in X\}$ where $[\mathcal{P}]_n(x)$ has been defined in Section A.3.

So we only have to count words ! Figure 6 summarizes Lemmas 6.6 and 6.7.

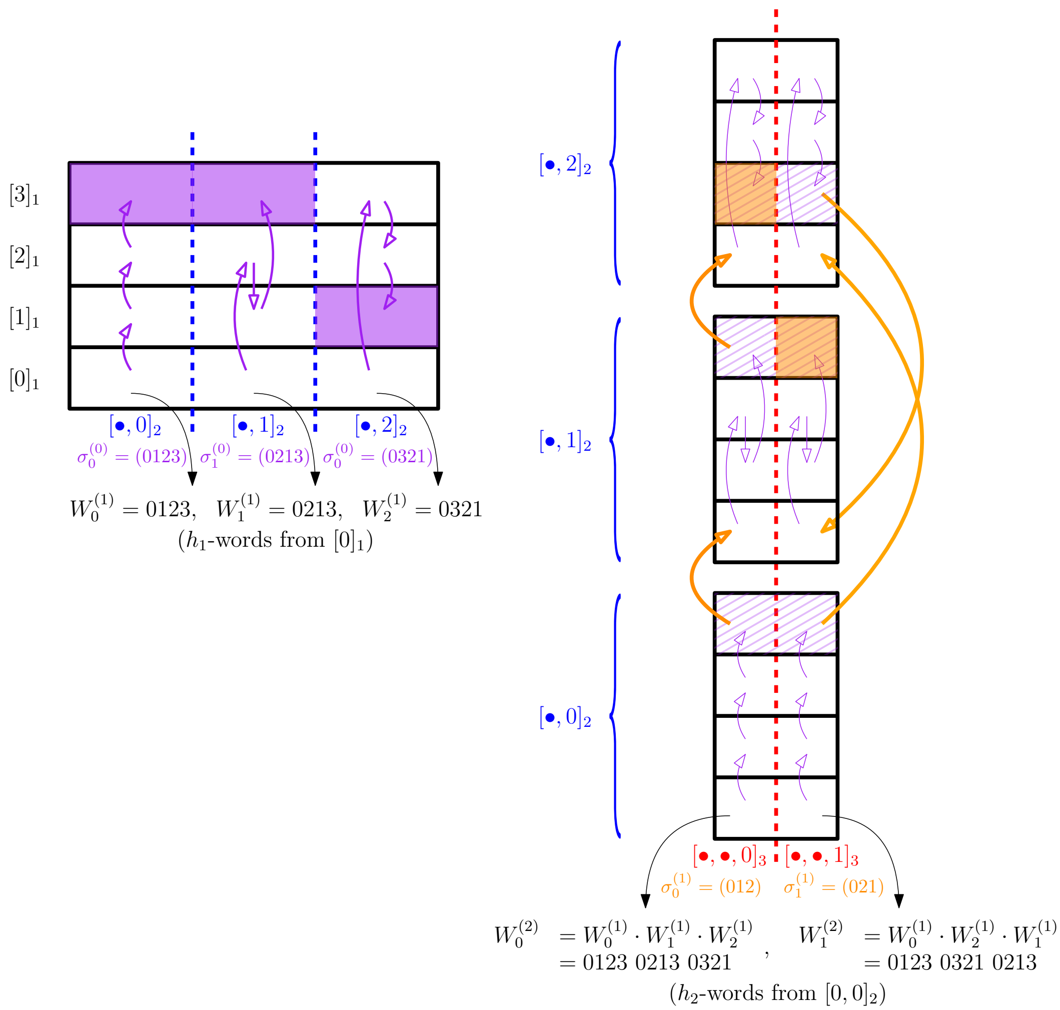


Figure 6: Words obtained from the bottom of the towers, with respect to the partition in 1-cylinders (a letter i in a word corresponds to the 1-cylinder $[i]_1$).

Lemma 6.6. *Let T be an odomutant built from an odometer S and permutations fixing 0. Let $\mathcal{P}$ be the partition in 1-cylinders. For every $n \geqslant 0$, for every $x_n \in \{0, 1, \ldots, q_n - 1\}$, the set*

$$\{[\mathcal{P}]_{h_n}(x) \mid x \in [0, \ldots, 0, x_n]_{n+1}\}$$

is a singleton, denoted by $\{\boldsymbol{W^{(n)}_{x_n}}\}$.

Proof of Lemma 6.6. Let $x \in [0, \ldots, 0, x_n]_{n+1}$. We can write $x = (\underbrace{0, \ldots, 0}_{n \text{ times}}, x_n, x_{n+1}, \ldots)$. All the permutations fix 0, so for every $i \geqslant n-1$, we have

$$\psi_i(x) = (\underbrace{0, \ldots, 0}_{n \text{ times}}, \sigma^{(n)}_{x_{n+1}}(x_n), \ldots, \sigma^{(i)}_{x_{i+1}}(x_i), x_{i+1}, x_{i+2}, \ldots).$$

For $k \in \{0, 1, \ldots, h_n - 1\}$, let $(k_0, k_1, \ldots, k_{n-1})$ be the n-tuple in $\prod_{0 \leqslant i \leqslant n-1} \{0, 1, \ldots, q_i - 1\}$ satisfying $k = k_0 + h_1 k_1 + \ldots + h_{n-1} k_{n-1}$. We then have

$$S^k \psi_i(x) = (k_0, k_1, \ldots, k_{n-1}, \sigma^{(n)}_{x_{n+1}}(x_n), \ldots, \sigma^{(i)}_{x_{i+1}}(x_i), x_{i+1}, x_{i+2}, \ldots)$$

so $T^k x$ is equal to $(y^{(k)}_0, \ldots, y^{(k)}_{n-1}, x_n, x_{n+1}, \ldots)$ where $y^{(k)}_i$ defined by

$$y^{(k)}_n = x_n,$$
$$\text{for } 0 \leqslant i \leqslant n-1 \quad y^{(k)}_i = \left(\sigma^{(i)}_{y^{(k)}_{i+1}}\right)^{-1}(k_i).$$

For every $k \in \{0, 1, \ldots, h_n - 1\}$, $(y^{(k)}_0, \ldots, y^{(k)}_{n-1})$ does not depend on $x_{n+1}, x_{n+2}, \ldots$ and only depends on x_n, so does the h_n-tuple $(y^{(k)}_0)_{0 \leqslant k \leqslant h_n - 1}$. The result follows from the fact that $[\mathcal{P}]_{h_n}(x)$ is equal to $(y^{(k)}_0)_{0 \leqslant k \leqslant h_n - 1}$. ☐

Lemma 6.7. *Let T be an odomutant built from an odometer S and permutations fixing 0. Let $\mathcal{P}$ be the partition in 1-cylinders, and recall the words $W^{(n)}_{x_n}$ defined in Lemma 6.6. For every $n \geqslant 1$ and $x_n \in \{0, 1, \ldots, q_n - 1\}$, we have*

$$W^{(n)}_{x_n} = W^{(n-1)}_0 \cdot W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(1)} \cdot \ldots \cdot W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(q_{n-1}-2)} \cdot W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(q_{n-1}-1)}.$$

Proof of Lemma 6.7. Given $n \geqslant 0$, note that we have

$$\{0, 1, \ldots, h_n - 1\} = \bigsqcup_{0 \leqslant i \leqslant q_{n-1}-1} \Big(\{0, 1, \ldots, h_{n-1} - 1\} + h_{n-1} i\Big).$$

Moreover if i is in $\{0, 1, \ldots, q_{n-1} - 1\}$, if x_n is in $\{0, 1, \ldots, q_n - 1\}$, we have

$$T^{h_{n-1} i}([0, \ldots, 0, 0, x_n]_{n+1}) = [0, \ldots, 0, \left(\sigma^{(n-1)}_{x_n}\right)^{-1}(i), x_n]_{n+1}.$$

This implies that, for a fixed $x \in [0, \ldots, 0, 0, x_n]_{n+1}$, the element $y_i := T^{h_{n-1} i}(x)$ is in $[0, \ldots, 0, \left(\sigma^{(n-1)}_{x_n}\right)^{-1}(i)]_n$ and we get

$$w_{h_{n-1} i, h_{n-1}(i+1)-1}(x) = w_{h_{n-1}}(T^{h_{n-1} i}(x)) = w_{h_{n-1}}(y_i) = W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(i)}$$

by Lemma 6.6. Finally the h_n-word on x is the following concatenation :

$$
\begin{aligned}
W^{(n)}_{x_n} &= w_{h_n}(x) \\
&= w_{0,h_n-1}(x) \\
&= w_{0,h_{n-1}-1}(x) \cdot w_{h_{n-1},2h_{n-1}-1}(x) \cdot \ldots \cdot w_{h_{n-1}(q_n-1),h_n-1}(x) \\
&= W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(0)} \cdot W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(1)} \cdot \ldots \cdot W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(q_{n-1}-2)} \cdot W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(q_{n-1}-1)} \\
&= W^{(n-1)}_{0} \cdot W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(1)} \cdot \ldots \cdot W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(q_{n-1}-2)} \cdot W^{(n-1)}_{\left(\sigma^{(n-1)}_{x_n}\right)^{-1}(q_{n-1}-1)}
\end{aligned}
$$

and we are done. □

6.4 Proof of the theorem

Lemma 6.8. *Same assumptions. We also assume that for every $n \geqslant 0$, the permutations $\sigma^{(n)}_0, \sigma^{(n)}_1, \ldots, \sigma^{(n)}_{q_{n+1}-1}$ are pairwise different. Then for every $n \geqslant 0$, the map*

$$x_n \in \{0, 1, \ldots, q_n - 1\} \mapsto W^{(n)}_{x_n}$$

is injective.

Proof of Lemma 6.8. Let us prove it by induction. The result is clear for $n = 0$. Given $n \geqslant 1$, assume that the map

$$x_{n-1} \in \{0, 1, \ldots, q_{n-1} - 1\} \mapsto W^{(n-1)}_{x_{n-1}}$$

is injective. By Lemma 6.7, this means that there exists a bijection between the set of words $\{W^{(n)}_{x_n} \mid x_n \in \{0, 1, \ldots, q_n - 1\}\}$ and the set of permutations $\{\sigma^{(n-1)}_{x_n} \mid x_n \in \{0, 1, \ldots, q_n - 1\}$, and the result follows from the fact that the map $x_n \mapsto \sigma^{(n-1)}_{x_n}$ is injective. □

Now let us assume that the map

$$\begin{cases} \{0, 1, \ldots, q_{n+1} - 1\} & \rightarrow & \{\sigma \in \mathrm{Sym}(\{0, 1, \ldots, q_n - 1\}) \mid \sigma(0) = 0 \text{ and } \sigma(q_n - 1) = q_n - 1\} \\ x_{n+1} & \mapsto & \sigma^{(n)}_{x_{n+1}} \end{cases}$$

is a bijection. In particular, $q_{n+1} = (q_n - 2)!$.

Let us introduce the sequence $\boldsymbol{(v_n)_{n\geqslant 0}}$ defined by

$$v_n = \frac{\log q_n}{h_n}.$$

Lemma 6.9. *The sequence $(v_n)_{n\geqslant 0}$ is bounded below by $v_0 - 6$. If q_0 is sufficiently large, then the odomutant T has positive topological entropy.*

Proof of Lemma 6.9. First we have

$$\log q_n = \log\left((q_{n-1} - 2)!\right) = \log\left((q_{n-1})!\right) - \log q_{n-1} - \log\left(q_{n-1} - 1\right) \geqslant \log\left((q_{n-1})!\right) - 2q_{n-1}.$$

Using $\log(k!) \geqslant k \log k - k$, we get

$$\log q_n \geqslant q_{n-1} \log q_{n-1} - 3q_{n-1}.$$

Using $h_n = h_{n-1} q_{n-1}$, we get

$$v_n \geqslant v_{n-1} - \frac{3}{h_{n-1}}.$$

Finally, we have the following lower bound

$$v_n \geqslant v_0 - 3 \sum_{i \geqslant 0} \frac{1}{h_i}.$$

The integer h_i is greater or equal to 2^i, so we get $v_n \geqslant v_0 - 6 = \log(q_0) - 6$.

By Lemma 6.8, we have

$$\frac{\log N\left((\mathcal{P})_0^{h_n - 1}\right)}{h_n} \geqslant \frac{\log q_n}{h_n} = v_n.$$

This implies $\mathrm{h}_{\mathrm{top}}(T) \geqslant \log(q_0) - 6$. If q_0 is sufficiently large, it gives $\mathrm{h}_{\mathrm{top}}(T) > 0$. □

Lemma 6.10. *The sequence (v_n) is bounded above.*

Proof of Lemma 6.10. Using $\log(k!) \leqslant k \log k$, we get

$$v_{n+1} = \frac{\log q_{n+1}}{h_n} \leqslant \frac{\log(q_n!)}{h_n} \leqslant \frac{q_n \log q_n}{h_n} = \frac{q_n \log q_n}{q_n h_{n-1}} = v_n.$$

Thus we have $v_n \leqslant v_0$ for every $n \geqslant 0$. □

Lemma 6.11. *For all integers $m \geqslant 0$, the sequence $\left(\frac{1}{\ln^{\circ m}(q_n)}\right)_{n \geqslant 0}$ is summable.*

Proof of Lemma 6.11. Let us consider an integer N such that

$$\forall n \geqslant N,\ (v_0 - 6) h_n \geqslant 1.$$

For every $n \geqslant N$, using Lemma 6.9, we have

$$\log q_{n+1} = h_{n+1} v_{n+1} \geqslant q_n \times (v_0 - 6) h_n \geqslant q_n.$$

By induction, we easily get for every $n \geqslant N$,

$$\log^{\circ m}(q_{n+m}) \geqslant q_n$$

The summability is now clear since the sequence $(q_n)_{n \geqslant 0}$ satisfies the relation $q_{n+1} = (q_n - 2)!$. □

Proof of Theorem B. With a sufficiently large integer q_0 such that the conclusion of Lemma 6.9 holds, and with the sequence $(q_n)_{n \geqslant 0}$ defined by the relation $q_{n+1} = (q_n - 2)!$, we have built an odomutant T of positive topological entropy from the odometer S on the space $\prod_{n \geqslant 0} \{0, 1, \ldots, q_n - 1\}$. By Proposition 6.4, the orbit equivalence built between T and S is a strong orbit equivalence. The recurrence relation on the sequence (q_n) implies that S is the universal odometer (every prime number p appear infinitely many time as a prime factor of the integers $q_0, q_1, \ldots$).

Finally, with the sequence $(h_n)_{n \geqslant 0}$ defined by $h_0 = 1$ and $h_{n+1} = h_n q_n$, the orbit equivalence is φ-integrable if $(\varphi(h_{n+1})/h_n)_n$ is summable, by Theorem 4.10 (see Condition (C2)). This holds for $\varphi(x) = \frac{\log(x)}{\log^{\circ m}(x)}$, since we have

$$\frac{\varphi(h_{n+1})}{h_n} = \frac{1}{\log^{\circ m}(h_{n+1})} \frac{\log(h_{n+1})}{h_n} \leqslant \frac{1}{\log^{\circ m}(q_n)} \left(\frac{\log(h_n)}{h_n} + v_n\right) \leqslant \frac{1}{\log^{\circ m}(q_n)}(1 + v_n)$$

and by Lemmas 6.11 and 6.10. □

7 On non-preservation of Kakutani equivalence under $\mathrm{L}^{<1/2}$ orbit equivalence

In this section, we sketch the proof of Theorem C that we recall:

Theorem 7.1 (C.)**.** *There exists an ergodic probability measure-preserving bijection T which is $\mathrm{L}^{<1/2}$ orbit equivalent (in particular Shannon orbit equivalent) to the dyadic odometer but not evenly Kakutani equivalent to it.*

Feldman [Fel76] has built a zero-entropy system which is not loosely Bernoulli. It turns out that we can describe this system as an odomutant built from the dyadic odometer, with parameters given by:

$$q_n = (2^{n+10})^{2^{n+12}+3},$$

and permutations that we will not specify. It is not difficult to check that the series $\sum \frac{\varphi(h_{n+1})}{h_n}$ converges for the map $\varphi(x) = x^p$ with $p < 1/2$, so the orbit equivalence is $\mathrm{L}^{<1/2}$ by Proposition 4.10.

Remark 7.2. In the construction of Feldman, the elements in $[0, \ldots, 0]_n$ produce h_n-words (with respect to the partition in 1-cylinders), describing the future, which are not pairwise f-close for the f-metric introduced in Section A.5 (therefore, the underlying system is not loosely Bernoulli). As an example, if the pieces of the partition are labelled with letters a and b, we want the elements of $[0, 0]_2$ to each describe h_1-words of the form

$$\begin{aligned} &ababab ab \ldots abababab, \\ &aabbaabb \ldots aabbaabb, \\ &aaaabbbb \ldots aaaabbbb, \ldots \end{aligned}$$

(note that they are not pairwise f-close) in such a manner that each $[0, 0, i]_2$ describes only one of these words and the (deterministic) laws for the futur cannot have pairwise good couplings. We continue this way for the second step, replacing letters by the new words built above.

A Background from ergodic theory

A.1 Ergodicity, mixing properties

Definition A.1. Let $T \in \mathrm{Aut}(X, \mu)$. We say that a measurable subset A of X is T-invariant if $\mu(A \Delta T(A)) = 0$.
T is ergodic if every T-invariant measurable subset is null or conull.

This is the definition of T-invariance in the simpler case of invertible transformations (when T is not invertible, A is T-invariant if $\mu(A \Delta T^{-1}(A)) = 0$).

If a set A satisfies $A = T(A)$, then for all points $x \in A$, its T-orbit is contained in A. This means that A is a union of orbits. In the case A is T-invariant, then A is a union of orbits up to zero measure. Therefore, ergodicity means that a measurable property only concerning the orbits is null or conull. For instance, given some measurable subset B, the property "there exist infinitely many positive integers n such that $T^n x$ lies in B" is satisfied for x if and only if it is satisfied for $T^k x$ for every $k \in \mathbb{Z}$, so it is a property on the orbits.

A stronger property than ergodicity is the weak mixing property.

Definition A.2. $T \in \mathrm{Aut}(X, \mu)$ is weakly mixing if for every measurable subsets A and B, the following holds:

$$\frac{1}{n} \sum_{i=0}^{n} \left| \mu(T^{-i}(A) \cap B) - \mu(A)\mu(B) \right| \underset{n \to +\infty}{\longrightarrow} 0.$$

Lemma A.3. *Strong mixing property is stronger than ergodicity.*

Proof. Let A be a T-invariant measurable set. Applying the weak mixing property to $B = A$, we get $\mu(A) = \mu(A)^2$, so $\mu(A)$ is either equal to 1 or to 0. □

Definition A.4. T is strongly mixing if for every measurable subsets A and B, the following holds:

$$\left| \mu(T^{-n}(A) \cap B) - \mu(A)\mu(B) \right| \underset{n \to +\infty}{\longrightarrow} 0.$$

Weak mixing property is a "Cesaro" version of strong mixing property, we deduce from this that strong mixing property is... stronger!

A.2 Point spectrum

Remark A.5. Given a measurable subset A and $T \in \mathrm{Aut}(X, \mu)$, we have $\mathbb{1}_A \circ T = \mathbb{1}_{T^{-1}(A)}$, so an equivalent definition of ergodicity is that every T-invariant characteristic function (seen as element of $\mathrm{L}^2(X, \mu)$) is either $\mathbb{1}_X$ or $\mathbb{1}$.

For every measurable subset C, we have $\mu(C) = \int_X \mathbb{1}_C \mathrm{d}\mu$. so the quantity $\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B)$ appearing in the definitions of weakly and strongly mixing can be written as

$$\int_X \left(\mathbb{1}_A \circ T^i\right) \mathbb{1}_B \mathrm{d}\mu - \int_X \mathbb{1}_A \mu \int_X \mathbb{1}_B \mu = \int_X \left(\mathbb{1}_A \circ T^i - \int_X \mathbb{1}_A \mu\right) \left(\mathbb{1}_B - \int_X \mathbb{1}_B \mu\right) \mathrm{d}\mu,$$

where the maps $\mathbb{1}_A \circ T^i - \int_X \mathbb{1}_A \mu$ and $\mathbb{1}_B - \int_X \mathbb{1}_B \mu$ have zero integral.

This remark leads us to the following functional viewpoint for all the properties introduced in the last section:

Definition A.6.
- T is ergodic if every T-invariant function of $\mathrm{L}^2(X,\mu)$ is constant almost everywhere.
- T is weakly wixing if for every $f, g \in \mathrm{L}^2(X,\mu)$ of zero integral, we have

$$\frac{1}{n}\sum_{i=0}^{n}\int_X \left(f\circ T^i\right) g\mathrm{d}\mu \underset{n\to+\infty}{\longrightarrow} 0$$

- T is strongly wixing if for every $f, g \in \mathrm{L}^2(X,\mu)$ of zero integral, we have

$$\int_X \left(f\circ T^n\right) g\mathrm{d}\mu \underset{n\to+\infty}{\longrightarrow} 0$$

Then it is natural study the unitary operator

$$U_T\colon f\in \mathrm{L}^2(X,\mu) \to f\circ T \in \mathrm{L}^2(X,\mu),$$

called the Koopman operator of T (it is unitary since μ is T-invariant). For example, we can look at its point spectrum.

Definition A.7. We say that $\lambda\in\mathbb{C}$ is an eigenvalue of T if there exists $f\in \mathrm{L}^2(X,\mu)\backslash\{0\}$ such that $f\circ T = \lambda f$. The function f is called an eigenfunction of T. Since U_T is unitary, λ is in the unit torus $\mathbb{T}$.

Example A.8. The constants functions are always eigenfunctions associated to the eigenvalue 1.

Lemma A.9. *Two eigenfunctions associated to distinct eigenvalues are orthogonal.*

Proof. Let $f_1, f_2 \in \mathrm{L}^2(X,\mu)$ be eigenfunctions respectively associated to the eigenvalues λ_1, λ_2. Using the fact that the Koopman operator is unitary, we get:

$$\langle f_1, f_2\rangle = \langle U_T f_1, U_T f_2\rangle = \langle \lambda_1 f_1, \lambda_2 f_2\rangle = \lambda_1\overline{\lambda_2}\langle f_1, f_2\rangle.$$

Since λ_1 and λ_2 are different points of the unit torus $\mathbb{U}$, we get $\lambda_1\overline{\lambda_2} \neq 1$, so $\langle f_1, f_2\rangle = 0$. ☐

Example A.10.
1. Let θ be an irrational number, and $R_\theta\colon z\in\mathbb{U}\mapsto z\exp(2i\pi\theta)\in\mathbb{U}$ be the irrational rotation of angle θ. For every $n\geqslant 0$, the map $z\in\mathbb{U}\to z^n\in\mathbb{U}$ is an eigenfunction of R_θ associated to the eigenvalue $\exp(2in\pi\theta)$. By Fourier analysis, the span of all the eigenfunctions of R_θ is dense in $\mathrm{L}^2(\mathbb{U})$, so the point spectrum of the R_θ is exactly $\{\exp(2in\pi\theta) \mid n\in\mathbb{Z}\}$.
2. Given a sequence $(q_n)_{n\geqslant 0}$ of integers greater than or equal to 2, and S the odometer on $X = \prod_{n\geqslant 0}\{0,1,\dots,q_n-1\}$, its point spectrum is exactly

$$\mathrm{Sp}(S) = \left\{\exp\left(\frac{2i\pi k}{h_n}\right) \mid n\geqslant 1, 0\leqslant k\leqslant h_n-1\right\}.$$

where $h_n \coloneqq q_0\dots q_{n-1}$. Indeed, it is straightforward to check that $f_\lambda\colon x\in X\mapsto \sum_{j=0}^{h_n-1}\lambda^j \mathbb{1}_{S^j([0,\dots,0]_n)}(x)$ is an eigenfunction associated to $\lambda = \exp\left(\frac{2i\pi k}{h_n}\right)$. Moreover, the span of all the eigenfunctions is dense in $\mathrm{L}^2(X)$, since a linear combination $\sum_{\ell=0}^{h_n-1} a_\ell f_{\lambda^\ell}$ is of the form $\sum_{j=0}^{h_n-1} P(\lambda^j)\mathbb{1}_{S^j([0,\dots,0]_n)}$ with the polynomial $P = a_0 + a_1 Y + \dots + a_{h_n-1}Y^{h_n-1}$, and good choices of P yield the characteristic functions of the n-cylinders.

Remark A.11. Since we can write $f = \left(f - \int_X f\mathrm{d}\mu\right) + \int_X f\mathrm{d}\mu$, the Hilbert space $\mathrm{L}^2(X,\mu)$ has the following orthogonal decomposition:

$$\mathrm{L}^2(X,\mu) = \mathrm{L}^2_0(X,\mu)\oplus\mathbb{C},$$

where $\mathrm{L}^2_0(X,\mu)$ and $\mathbb{C}$ are respectively the closed subspaces of zero-integral functions and constant functions. These subspaces are stabilized by U_T. Since the constant functions are always eigenfunctions of T (associated to the eigenvalue 1), it is usually relevant to only study the restriction of U_T on $\mathrm{L}^2_0(X,\mu)$ (Koopman reduced operator).

We can thus reformulate some dynamical properties this way:

Proposition A.12.
- *T is ergodic if and only if the constant functions are the only eigenfunctions associated to the eigenvalue 1 (equivalently, the eigenvalue 1 is simple).*
- *If T is ergodic, then it is weakly mixing if 1 is the only eigenvalue.*

Corollary A.13. *Odometers are not weakly mixing.*

Let us define a particular class of systems.

Definition A.14. A system has discrete spectrum if the span of all its eigenfunctions is dense in $\mathrm{L}^2(X, \mathcal{A}, \mu)$.

Example A.15. In Example A.10, we got two examples of discrete-spectrum systems:

1. irrational rotations (by Fourier analysis on the unit torus $\mathbb{U}$);
2. odometers.

The following result, due to Halmos and von Neumann, provides a classification of ergodic discrete-spectrum systems up to conjugacy.

Theorem A.16 (Halmos, von Neumann [HVN42])**.** *Two ergodic systems of discrete spectrum are conjugate if and only if they have the same point spectrum.*

This is also a classification up to flip-conjugacy since the point spectrum is symmetric.

A.3 Measurable partitions, entropy

Measurable partition A set $\mathcal{P}$ of measurable subsets of X is a **measurable partition** of X if:

- for every $P_1, P_2 \in \mathcal{P}$, we have $\mu(P_1 \cap P_2) = 0$;
- the union $\bigcup_{P \in \mathcal{P}} P$ has full measure.

The elements of $\mathcal{P}$ are called the **atoms**. If $\mathcal{P}$ and $\mathcal{Q}$ are measurable partitions of (X, μ), we say that $\mathcal{P}$ **refines** (or is a refinement of, or is finer than) $\mathcal{Q}$, denoted by $\mathcal{P} \succcurlyeq \mathcal{Q}$, if every atom of $\mathcal{Q}$ is a union of atoms of $\mathcal{P}$ (up to a null set). More generally, their **joint partition** is

$$\mathcal{P} \vee \mathcal{Q} \coloneqq \{P \cap Q \mid P \in \mathcal{P}, Q \in \mathcal{Q}\},$$

namely the least fine partition which refines $\mathcal{P}$ and $\mathcal{Q}$. This operation $\vee$ is associative.

A measurable partition $\mathcal{P}$ defines almost everywhere a map $\mathcal{P}(.)\colon X \to \mathcal{P}$ where $\mathcal{P}(x)$ is the atom of $\mathcal{P}$ which contains x. Given a measurable map $T\colon X \to X$, $\mathcal{P}$ provides **coding maps**

$$[\mathcal{P}]_{i,n}\colon x \in X \mapsto (\mathcal{P}(T^j x))_{i \leqslant j \leqslant n} \in \mathcal{P}^{\{i,\dots,n\}}.$$

In particular, $[\mathcal{P}]_n(x) \coloneqq [\mathcal{P}]_{0,n-1}(x)$ is the n**-word** of x.

Given atoms $P_i, P_{i+1}, \dots, P_n$ of $\mathcal{P}$, the equality $[\mathcal{P}]_{i,n}(x) = (P_i, \dots, P_n)$ exactly means that x is an element of $T^{-i}(P_i) \cap T^{-(i+1)}(P_{i+1}) \cap \dots \cap T^{-n}(P_n)$. Therefore the partition which gives the values of $[\mathcal{P}]_{i,n}$ is the following joint partition

$$\mathcal{P}_i^n \coloneqq \bigvee_{j=i}^{n} T^{-j}(\mathcal{P})$$

with $T^{-j}(\mathcal{P}) \coloneqq \{T^{-j}(P) \mid P \in \mathcal{P}\}$, this is a division of the space given by the dynamic of T, over the timeline $\{i, \dots, n\}$ and with respect to $\mathcal{P}$.

Entropy of a partition If $\mathcal{P} = \{P_1, P_2\}$ is a partition of X with two atoms of equal measure, and $\mathcal{Q} = \{Q_1, Q_2\}$ is another partition of cardinality 2, such that $\mu(Q_1) = 0,9999$, then $\mathcal{P}$ brings more information than $\mathcal{Q}$. Given a random variable x with law μ, the answer to the question "In which atom is x ?" is of more interest for $\mathcal{P}$ since it is uncertain. We want a function (called entropy) from the set of measurable partitions to $\mathbb{R}_+$ which quantify the uncertainty of the answer, or how much a partition divides the space.

First, we define the information function I from the measurable function to $\mathbb{R}_+$, such that, given a random variable x with law μ, $I(P)$ quantifies how much it is surprising to find out that x lies in P. We heuristically get the following axioms:

- $I(X) = 0$;
- $I(\varnothing) = +\infty$;
- $I(P) = f(\mu(P))$ for a decreasing map f;
- if A and B are independant, then $I(A \cap B) = I(A) + I(B)$.

The map f is necessarily $-\log$ (up to a multiplicative constant), so we define

$$I(P) \coloneqq -\log \mu(P).$$

Then we define the information function of a partition $\mathcal{P}$ as

$$I_{\mathcal{P}} \coloneqq \sum_{P \in \mathcal{P}} I(P) \mathbb{1}_P \colon X \to \mathbb{R}_+$$

and the entropy is the mean of this function:

$$H_\mu(\mathcal{P}) \coloneqq \int_X I_{\mathcal{P}}(x) \mathrm{d}\mu(x) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P) \in \mathbb{R}_+.$$

A well-known result states that, given an positive integer n, the maximum of H_μ on the set of partitions of cardinality n is reached on the uniform partitions, so this is the notion of entropy we were looking for at the beginning of the paragraph.

A.4 Measure-theoretic entropy, topological entropy

Here we present two notions of entropy. For more details, the reader may refer to [Dow11] and [KL16].

Measure-theoretic entropy. Entropy, or measure-theoretic entropy, or metric entropy, of a measurable transformation is an invariant of conjugacy. To define it, we first define the entropy of a partition, which then enables us to quantify how much a transformation complexifies the partitions.

Let T be a system on (X, μ), not necessarily invertible, and $\mathcal{P}$ a finite measurable partition of X. The following quantity

$$\mathrm{h}_\mu(T, \mathcal{P}) \coloneqq \lim_{n \to +\infty} \frac{\mathrm{H}_\mu(\mathcal{P}_0^{n-1})}{n}$$

is well-defined, this is the **entropy of T with respect to $\mathcal{P}$**, and it tells us how quickly the dynamic of T is dividing the space X with the partition $\mathcal{P}$. Finally, let us define the **entropy of T** by

$$\mathrm{h}_\mu(T) \coloneqq \sup_{\mathcal{P}} \mathrm{h}_\mu(T, \mathcal{P}),$$

where the supremum is over all the finite measurable partitions $\mathcal{P}$ of X. This quantity is non-negative and can be infinite.

The following result enables us to prove that the odometers have zero entropy (see Proposition 3.5).

Proposition A.17. *If $(\mathcal{P}_k)_{k \geqslant 0}$ is a sequence of partitions which increases to the σ-algebra of X, then we have*

$$\mathrm{h}_\mu(T, \mathcal{P}_k) \underset{k \to +\infty}{\to} \mathrm{h}_\mu(T).$$

Topological entropy. In the topological setting, topological entropy is an invariant of topological conjugacy and is defined with similar ideas.

The topological space X has to be compact. We define the **joint cover** of two open covers $\mathcal{U}$ and $\mathcal{V}$ by

$$\mathcal{U} \vee \mathcal{V} := \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}.$$

This is an associative operation. Let T be a topological system on X and $\mathcal{U}$ an open cover of X. Let us define

$$\mathcal{U}_0^{n-1} := \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{U}),$$

where $T^{-i}(\mathcal{U}) := \{T^{-i}(U) \mid U \in \mathcal{U}\}$, and

$$\mathcal{N}(\mathcal{U}) := \min\{|\mathcal{U}'| \mid \mathcal{U}' \text{ is a subcover of } \mathcal{U}\},$$

where $|\mathcal{U}'|$ denotes the cardinality of $\mathcal{U}'$. The quantity $\mathcal{N}(\mathcal{U})$ is finite since X is compact.

The **topological entropy of T with respect to the open cover $\mathcal{U}$** is the well-defined limit

$$\mathrm{h_{top}}(T, \mathcal{U}) := \lim_{n \to +\infty} \frac{\log \mathcal{N}(\mathcal{U}_0^{n-1})}{n},$$

it tells us how quickly the dynamic of T is shrinking the open sets of $\mathcal{U}$.

Finally, let us define the **topological entropy of T** by

$$\mathrm{h_{top}}(T) := \sup_{\mathcal{U}} \mathrm{h_{top}}(T, \mathcal{U}),$$

where the supremum is over all the open covers $\mathcal{U}$ of X. This quantity is non-negative and can be infinite.

We say that a sequence $(\mathcal{U}_n)_{n \geqslant 0}$ of open covers generates the topology on X if for every $\varepsilon > 0$, there exists $N \geqslant 0$ such that for every $n \geqslant N$, the open sets of $\mathcal{U}_n$ has a diameter less than ε. The following statement is an topological version of Proposition A.17.

Proposition A.18. *Let T be a topological system on X and $(\mathcal{U}_n)_{n \geqslant 0}$ a generating sequence of open covers. Then we have*

$$\mathrm{h_{top}}(T) = \lim_{n \to +\infty} \mathrm{h_{top}}(T, \mathcal{U}_n).$$

Example A.19. The compact space X that we consider in these notes is of the form

$$X := \prod_{n \geqslant 0} \{0, 1, \dots, q_n - 1\},$$

with integers q_n greater or equal to 2. It admits open covers which are partitions in clopen sets. If $\mathcal{U}$ is such an open cover, then $\mathcal{U}_0^{n-1}$ denotes both joint of open covers and joint of partitions. We have $\mathcal{N}(\mathcal{U}_0^{n-1}) = |\mathcal{U}_0^{n-1} \backslash \{\varnothing\}|$ and this is exactly the number of words of the form $[\mathcal{U}]_n(x)$, for $x \in X$, where $[\mathcal{U}]_n$ is the coding map associated to the partition $\mathcal{U}$ (see Section A.3). Therefore, in the proof of Theorem B, a method to create topological entropy consists in building a system T whose number of n-words (with respect to some partition in clopen sets) increases quickly enough as n goes to ∞.

The variational principle. In Example A.19, we explain the method that we will apply in this paper to create topological entropy and then prove Theorem B (or more generally Theorem 2.16). However we also would like to prove the same statement in a measure-theoretical setting (see Theorem 2.15). The variational principle enables us to connect topological and measure-theoretical entropies, and to get Theorem 2.15 as a Corollary of Theorem 2.16 (if unique ergodicity holds).

Theorem (Variational principle)**.** *Let $T \colon X \to X$ be a topological system on a metric compact set X. Then we have*

$$\mathrm{h_{top}}(T) = \sup_{\mu} \mathrm{h}_{\mu}(T)$$

where the supremum is over all T-invariant Borel probability measures μ on X.

As a consequence, if T is uniquely ergodic, then we have

$$\mathrm{h}_{\mathrm{top}}(T) = \mathrm{h}_\mu(T),$$

where μ denotes the only T-invariant Borel probability measure.

A.5 Even Kakutani equivalence, loose Bernoullicity

The notions introduced in this section can be found in [Fel76] and [ORW82].

Let $T \in \mathrm{Aut}(X, \mu)$. Given a measurable set A, the return time $r_A \colon A \to \mathbb{N}^* \cup \{\infty\}$ is defined by:

$$\forall x \in A,\ r_A(x) \coloneqq \inf\{k \geqslant 1 \mid T^k x \in A\}.$$

It follows from Poincaré recurrence theorem that, if A has positive measure, then the set $\{k \in \mathbb{N}^* \mid T^k x \in A\}$ is infinite for almost every $x \in A$. In particular, $r_A(x)$ is finite for almost every $x \in A$.

Then we can define a transformation T_A on the set $\{x \in A \mid r_A(x) < \infty\}$, namely on A up to a null set, called the **induced tranformation** on A:

$$T_A x \coloneqq T^{r_A(x)} x.$$

The map T_A is an element of $\mathrm{Aut}(A, \mu_A)$, where $\mu_A \coloneqq \mu(.)/\mu(A)$ is the conditional probability measure. Its entropy is given by Abramov's formula:

$$\mathrm{h}_{\mu_A}(T_A) = \frac{\mathrm{h}_\mu(T)}{\mu(A)}.$$

Definition A.20. Let $S \in \mathrm{Aut}(X, \mu)$, $T \in \mathrm{Aut}(Y, \nu)$ be two ergodic transformations.

1. T and S are said to be **Kakutani equivalent** if T_A and S_B are isomorphic for some measurable sets $A \subset X$ and $B \subset Y$.
2. Moreover they are **evenly Kakutani equivalent** if in addition two such measurable sets have the same measure: $\mu(A) = \nu(B)$.

It is well-known that Kakutani equivalence and even Kakutani equivalence are equivalence relations. It follows from Abramov's formula that entropy is preserved under even Kakutani equivalence.

Similarly to Ornstein's theory [Orn70] for the conjugacy problem, Ornstein, Rudolph and Weiss [ORW82] found a class of systems, called loosely Bernoulli system, where Kakutani and even Kakutani equivalences are well understood. These systems were first introduced by Feldman [Fel76].

Definition A.21 (see [Fel76]).

- The f-metric between words of same length is defined by:

$$f_n((a_i)_{1 \leqslant i \leqslant n}, (b_i)_{1 \leqslant i \leqslant n}) = 1 - \frac{k}{n}$$

where k is the greatest integer for which we can find equal subsequences $(a_{i_\ell})_{1 \leqslant \ell \leqslant k}$ and $(b_{j_\ell})_{1 \leqslant \ell \leqslant k}$, with $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$.

- Let $T \in \mathrm{Aut}(X, \mu)$ and $\mathcal{P}$ be a partition of X. The couple $(T, \mathcal{P})$, called a process, is **loosely Bernoulli** if for every $\varepsilon > 0$, for every sufficiently large integer N and for each $M > 0$, there exists a collection $\mathcal{G}$ of "good" atoms in $\mathcal{P}_{-M}^0$ whose union has measure greater than $1 - \varepsilon$, and so that for each pair A, B of atoms in $\mathcal{G}$, the following holds: there is a measure $n_{A,B}$ on $\mathcal{P}^N \times \mathcal{P}^N$ satisfying
  1. $n_{A,B}(\{P\} \times \mathcal{P}^N) = \mu_A(\{[\mathcal{P}]_{1,N}(.) = P\})$ for every $P \in \mathcal{P}^N$;
  2. $n_{A,B}(\mathcal{P}^N \times \{P\}) = \mu_B(\{[\mathcal{P}]_{1,N}(.) = P\})$ for every $P \in \mathcal{P}^N$;
  3. $n_{A,B}(\{(P, P') \in \mathcal{P}^N \times \mathcal{P}^N \mid f_N(P, P') > \varepsilon\}) < \varepsilon$.
- T is **loosely Bernoulli** if $(T, \mathcal{P})$ is loosely Bernoulli for all finite partitions $\mathcal{P}$ of X.

Loose Bernoullicity for a process $(T, \mathcal{P})$ expresses the fact that, conditionally to two pasts A and B, the laws for the future are close, meaning that there exists a good coupling between them, with close words for the f-metric.

Proposition A.22. *Let $T \in \mathrm{Aut}(X, \mu)$.*

1. *If $\mathcal{P}$ is a generating partition and if $(T, \mathcal{P})$ is loosely Bernoulli, then T is loosely Bernoulli.*
2. *If $(\mathcal{P}_k)$ is a sequence of partitions increasing to the σ-algebra of X, and if $(T, \mathcal{P}_k)$ is loosely Bernoulli for every k, then T is loosely Bernoulli.*

Example A.23.

1. The Bernoulli shift on $\{1, \ldots, k\}^{\mathbb{Z}}$ is loosely Bernoulli with respect to the partition $\{[1]_1, \ldots, [k]_1\}$. Indeed, conditionally to every past, the law for the N-word is always the uniform distribution on $\{1, \ldots, k\}^N$. This system is more generally loosely Bernoulli since $\{[1]_1, \ldots, [k]_1\}$ is a generating partition.
2. Odometers are loosely Bernoulli (see Proposition 3.6).

The choice of the metric is very important. Indeed, with the d-metric:

$$d_n((a_i)_{1 \leqslant i \leqslant n}, (b_i)_{1 \leqslant i \leqslant n}) = |\{1 \leqslant i \leqslant n \mid a_i \neq b_i\}|,$$

also called the Hamming distance, we get the notion of very weakly Bernoulli systems and this is exactly the class considered in Ornstein's theory for the conjugacy problem.

As mentioned above, Kakutani equivalence and even Kakutani equivalence are well understood in the class of loosely Bernoulli systems.

Theorem (see Theorems 5.1 and 5.2 in [ORW82])**.** *Let $S \in \mathrm{Aut}(X, \mu)$, $T \in \mathrm{Aut}(Y, \nu)$ be two ergodic transformations.*

1. *If S is loosely Bernoulli and is Kakutani equivalent to T, then T is also loosely Bernoulli.*
2. *If S and T are loosely Bernoulli, then they are evenly Kakutani equivalent if and only if they have the same entropy.*

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