Introduction to Kakutani equivalence

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These notes aim to introduce the notion of Kakutani equivalence, its generalization to $\mathbb{Z}^d$-actions and a classification theory.

$(X, \mathcal{A}, \mu)$ will always denote a standard Borel probability space and $\mathrm{Aut}(X, \mathcal{A}, \mu)$ the set of bimeasurable probability measure preserving maps $T : X \to X$ (two such maps being identified if they coincide on a measurable set of full measure).

Every $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$ gives a $\mathbb{Z}$-action via $(n, x) \mapsto T^n x$. Later on we will more generally consider p.m.p. group actions $T : G \curvearrowright (X, \mathcal{A}, \mu)$, more precisely for $G = \mathbb{Z}^d$.

Contents

1 Induced map, Kakutani tower

The transformations will always be invertible as the proofs are easier and we will consider more generally group actions later on. However the notions and properties in this section are valid for non invertible transformations.

1.1 Induced map

Let $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$ and $A \in \mathcal{A}$ of positive measure. The return time $r_{T,A} : A \to \mathbb{N}^* \cup \{\infty\}$ is defined by :

$$\forall x \in A,\ r_{T,A}(x) := \inf \{k \geqslant 1 \mid T^k x \in A\},$$

also written r_A if the context is clear.

Theorem 1.1 (Poincaré recurrence theorem)**.** *If $\mu(A) > 0$, then for almost every $x \in A$, the set $\{k \in \mathbb{N}^* \mid T^k x \in A\}$ is infinite.*

When T is ergodic, Theorem 1.1 is true for almost every x in X. Indeed by this theorem it suffices to show that almost every orbit visits A, i.e. to show that $\bigcup_{n\in\mathbb{Z}} T^n(A)$ is of full measure, but this set is T-invariant and of positive measure since it contains A, then the result follows.

Proof of Theorem 1.1. For every $n \in \mathbb{N} \cup \{\infty\}$, we define

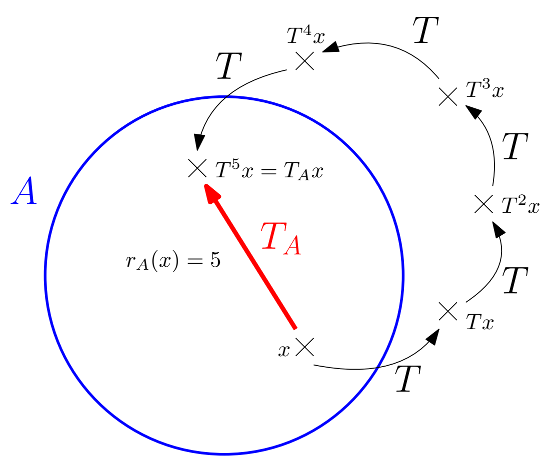
$$E_n := \{x \in A \mid \sup\{k \geqslant 0 \mid T^k x \in A\} = n\}.$$

For every finite n, we have $T^n(E_n) = E_0$ so E_n has the same measure as E_0, but this measure is necessary zero since A is the disjoint union of the E_n for $n \in \mathbb{N} \cup \{\infty\}$ and μ is finite. Thus A is equal to E_∞ up to a null set. □

Then we can define a transformation T_A on the set of x given by Poincaré recurrence theorem, i.e. on A up to a null set, called the induced tranformation on A :

$$T_A x := T^{r_A(x)} x.$$

T_A is an element of $\mathrm{Aut}(A, \mathcal{A}_A, \mu_A)$, where $\mathcal{B}_A$ is the set of elements of $\mathcal{B}$ included in A and $\mu_A = \mu(.)/\mu(A)$ is the conditional probability measure. Indeed every subset B of A is the disjoint union of the $B_n := B \cap \{r_A = n\}$ and $T_A(B_n) = T^n(B)$ by definition, then the equality $\mu_A(T_A(B)) = \mu_A(B)$ follows from the T-invariance of μ.



A consequence of the next lemma is that r_A is μ_A-integrable.

Theorem 1.2 (Kac's theorem)**.**

$$\int_A r_A \mathrm{d}\mu_A \leqslant \frac{1}{\mu(A)}$$

with equality when T is ergodic.

Proof of Theorem 1.2. X contains the disjoint union $\bigcup_{k>0}\bigcup_{0\leqslant n<k} T^n(\{r_A = k\})$ (with equality up to a null set when T is ergodic). Considering the measure of both sets, this gives the desired formula. □

In this proof, we considered the subsets $T^n(\{r_A = k\})$ for $0 \leqslant n < k$. When one wants to recover T from T_A and r_A, the goal is exactly to artificially build these subsets (see the next part about Kakutani tower).

Moreover ergodicity is preserved by induction.

Proposition 1.3. *If T is ergodic, then so is T_A.*

Proof of Proposition 1.3. Consider a T_A-invariant subset B of A and $B' := \bigcup_{n\in\mathbb{N}} T^n(B)$ which is also $\bigcup_{k\in\mathbb{N}}\bigcup_{0\leqslant n<k} T^n(\{x \in B \mid r_A(x) = k\})$. Both equivalent definitions imply that B' is T-invariant and $B' \cap A$ is equal to B. Then by the ergodicity of T, $\mu(B')$ is zero or one, and $\mu_A(B)$ too. □

For $x \in A$, $T_A x$ is obtained by mapping $r_A(x)$ times the transformation T on x. Then, to recover T from T_A and r_A, we have to create $r_A(x)$ virtual steps between x and $T_A x$. In order to do so, A will be enlarged, setting a second argument which will be incremented by the new transformation before applying T_A to the point of A. This is called a Kakutani tower of height r_A built over A. The definition of this transformation is given in the next part.

1.2 Kakutani tower

Let $S \in \mathrm{Aut}(Y, \mathcal{B}, \nu)$ and $h\colon Y \to \mathbb{N}^*$ be an integrable function. We define a new space

$$Y^h := \{(y, i) \mid y \in Y, 0 \leqslant i < h(y)\},$$

it is the disjoint union of the

$$Y_i^j := \{h = j\} \times \{i\}$$

for $0 \leqslant i < j$. If $h' \leqslant h$, then $Y^{h'}$ is included in Y^h. Moreover the subset $Y \times \{0\}$ of Y^h is exactly Y^1 (ie $Y^{h'}$ with $h' = 1$), it is a copy of Y. Notice that Y_i^j is a copy of the subset $\{h = j\}$ of Y via the bijection $f_i^j : y \in \{h = j\} \mapsto (y, i) \in Y_i^j$, then Y^h is endowed with a natural σ-algebra $\mathcal{B}^h$ and a natural probability measure ν^h.

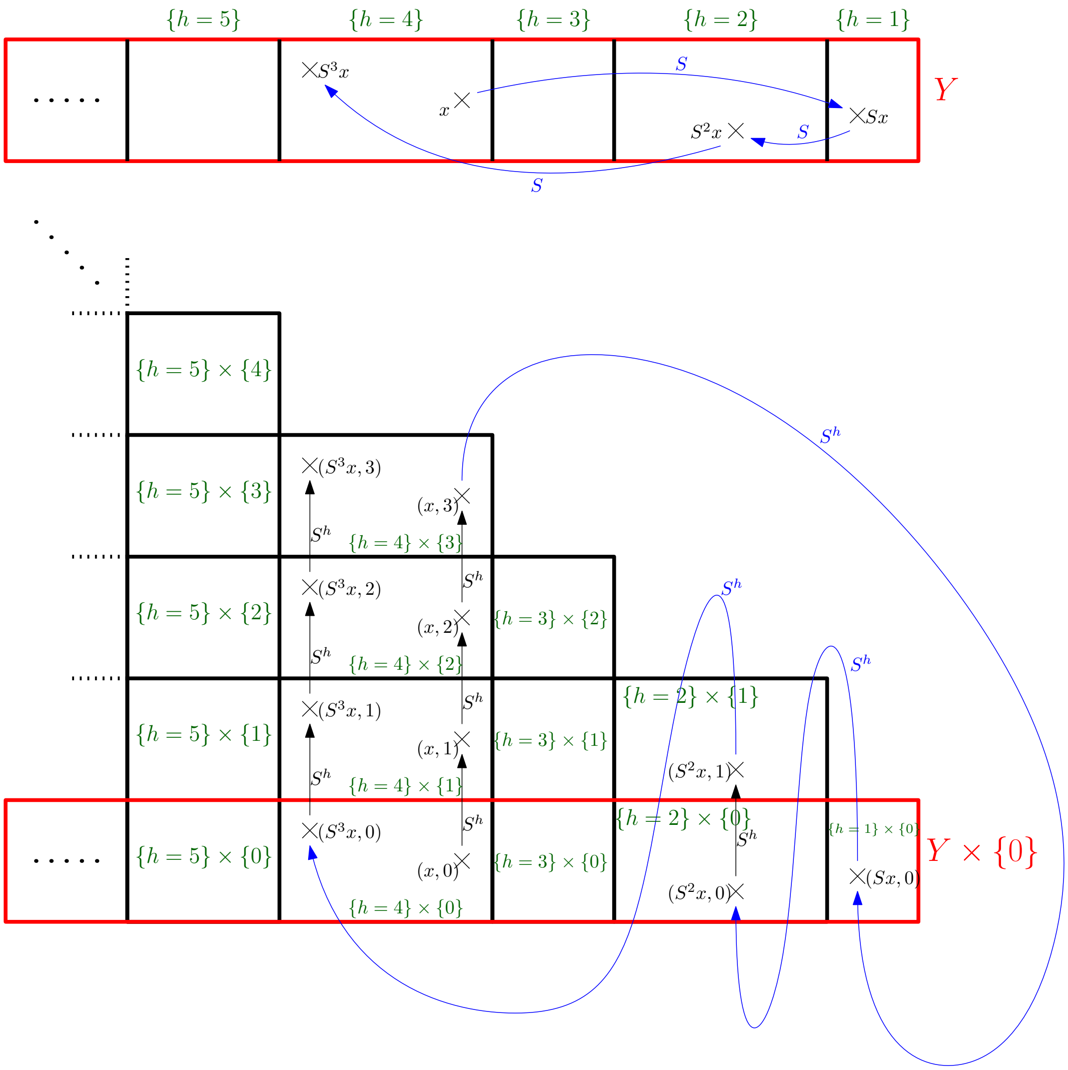
- $\mathcal{B}^h$ is the product σ-algebra of $Y \times \mathbb{N}$ restricted to Y^h. In particular, for all measurable set B included in $\{h = j\}$, $B \times \{i\}$ is a measurable set in Y^h for every $0 \leqslant i < j$, and $(\mathcal{B}^h)_{Y^1} = \{B \times \{0\} \mid B \in \mathcal{B}\}$.

- ν^h is within a renormalization constant the product measure $\nu \otimes \delta_{\mathbb{N}}$ restricted to $\mathcal{B}^h$, where $\delta_{\mathbb{N}}$ is the counting measure on $\mathbb{N}$. Intuitively ν^h behaves on each Y_i^j as ν does on $\{h = j\}$, meaning that for every measurable set B included in Y_i^j, $\nu^h(B) := \lambda \times \nu((f_i^j)^{-1}(B))$ where λ is a renormalization constant. Such a constant exists, i.e. ν^h is a finite measure, since h is integrable (see the computation of λ at the end of this part).

Now the transformation S^h acting on Y^h is defined by :

$$S^h(y,i) := \begin{cases} (y, i+1) & \text{if } i+1 < h(y) \\ (Sy, 0) & \text{if } i+1 = h(y) \end{cases}$$

It is an element of $\mathrm{Aut}(Y^h, \mathcal{B}^h, \nu^h)$. (Y^h, S^h) is called a tower of height h over Y.



Proposition 1.4. *If S is ergodic, then so is S^h.*

Proof of Proposition 1.4. Consider a S^h-invariant subset B of Y^h and $B' := B \cap Y^1$. B' is S-invariant and B is the union of the $(S^h)^n(B')$ for $n \in \mathbb{Z}$. Then B is trivial by ergodicity of S. □

Coming back to the notion of induced maps, it is not difficult to see that towers are the inverse operation, in the sense that

- when T is ergodic[1], $(A^{r_A}, (T_A)^{r_A})$ is isomorphic to (X, T) (here it is a tower with $Y = A$, $\nu = \mu_A$ and $h = r_A \colon A \to \mathbb{N}^*$ which is of μ_A-integral $1/\mu(A)$ by Kac's theorem);

- $r_{S^h, Y^1} = h$ and $(Y^1, (S^h)_{Y^1})$ is isomorphic to (Y, S).

It is interesting to compute the value of λ.

$$1 = \sum_{0 \leqslant i < j} \nu^h(Y_i^j) = \lambda \sum_{0 \leqslant i < j} \nu(\{h = j\}) = \lambda \sum_{0 < j} j\nu(\{h = j\}) = \lambda \int_Y h \mathrm{d}\nu.$$

Then λ is the inverse of the integral of h and it is also the ν^h-measure of Y^1 since $\nu(Y) = 1$.

Then notice that the less is the measure of the induction subset $A \subset X$, the greater will be the integral of the height function defined on A to recover (X, T). Conversely, the greater is the integral of the height function $h \colon Y \to \mathbb{N}^*$, the less will be the measure of the induction subset Y^1 of Y^h to recover (Y, S).

1.3 Properties of induced maps and towers

Proposition 1.5 (behaviour with conjugacy). *Let $S \in \mathrm{Aut}(Y, \mathcal{B}, \nu)$ and φ a measure isomorphism from $(X, \mathcal{A}, \mu)$ to $(Y, \mathcal{B}, \nu)$.*

1. *For every $A \subset X$,*
$$(\varphi^{-1} S \varphi)_A = \varphi^{-1} S_{\varphi(A)} \varphi.$$

2. *For every integrable function $h \colon X \to \mathbb{N}^*$,*
$$(\varphi^{-1} S \varphi)^h = \psi^{-1} S^{h \circ \varphi^{-1}} \psi$$
with $\psi \colon (x, i) \in X^h \to (\varphi(x), i) \in Y^h$.

Remark 1.6. The first statement of the last proposition tells us, in some sense, that after inducing two isomorphic transformations on the "same" set, the isomorphism remains true between the induced maps. Indeed, consider a conjugation $\varphi \colon X \to Y$ between $T \in (X, \mathcal{A}, \mu)$ and $S \in (Y, \mathcal{B}, \nu)$, and $A \subset X$. Then
$$T_C = (\varphi^{-1} S \varphi)_C = \varphi^{-1} S_{\varphi(C)} \varphi.$$
Inducing one transformation on $C \subset X$ is equivalent to inducing the other on the corresponding subset $\varphi(C)$ of Y given by the conjugation φ.

It is the same idea for the second statement : a tower of height h for T is like a tower of height $h \circ \varphi^{-1}$ (the corresponding height defined on Y, given by the conjugation φ) for S.

[1] Without ergodicity, one only recovers the system $(X', T \colon X' \to X')$ with $X' = \bigsqcup_{k>0} \bigsqcup_{0 \leqslant n < k} T^n(\{r_A = k\})$ (see the proof of Theorem 1.2).

Proof of Proposition 1.5. 1. For $x \in A$, we have

$$\begin{aligned} r_{\varphi^{-1}S\varphi,A}(x) &= \min\{n>0 \mid \varphi^{-1}S^n\varphi(x) \in A\} \\ &= \min\{n>0 \mid S^n\varphi(x) \in \varphi(A)\} \\ &= r_{S,\varphi(A)}(\varphi(x)). \end{aligned}$$

Then

$$(\varphi^{-1}S\varphi)_A x = (\varphi^{-1}S\varphi)^{r_{\varphi^{-1}S\varphi,A}(x)} x = \varphi^{-1}S^{r_{S,\varphi(A)}(\varphi(x))}\varphi(x) = \varphi^{-1}S_{\varphi(A)}(\varphi(x)).$$

2. Let $x \in X$. If $0 \leqslant i < h(x)-2$, then $(\varphi^{-1}S\varphi)^h(x,i) = (x,i+1)$. Moreover we have $0 < i < h \circ \varphi^{-1}(\varphi(x)) - 2$, so $S^{h\circ\varphi^{-1}}(\varphi(x),i) = (\varphi(x),i+1)$, this gives

$$\psi^{-1}S^{h\circ\varphi^{-1}}\psi(x,i) = \psi^{-1}(\varphi(x),i+1) = (x,i+1).$$

If $i = h(x)-1$, then $(\varphi^{-1}S\varphi)^h(x,i) = (\varphi^{-1}S\varphi(x),0)$. Moreover we have $i = h\circ\varphi^{-1}(\varphi(x)) - 1$, so $S^{h\circ\varphi^{-1}}(\varphi(x),i) = (S\varphi(x),0)$, this gives

$$\psi^{-1}S^{h\circ\varphi^{-1}}\psi(x,i) = \psi^{-1}(S\varphi(x),0) = (\varphi^{-1}S\varphi(x),0).$$

□

Proposition 1.7 (smaller and smaller induction subsets, greater and greater height functions)**.** *Let $T \in \mathrm{Aut}(X,\mathcal{A},\mu)$.*

1. *If $A_1 \subset A_2 \subset X$, then $(T_{A_2})_{A_1} = T_{A_1}$ and*

$$r_{T,A_1}(x) = \sum_{0 \leqslant i < r_{T_{A_2},A_1}(x)} r_{T,A_2}(T^i_{A_2}(x)).$$

2. *If $h_1, h_2 \colon X \to \mathbb{N}^*$ are two integrable functions and $h_1 \leqslant h_2$ on X, then $T^{h_2} = (T^{h_1})^g$ with*

$$g(x,i) := \begin{cases} 1 & \text{if } i < h_1(x) \\ h_2(x) - h_1(x) + 1 & \text{if } i = h_1(x) \end{cases}.$$ [2]

Moreover

$$\int_{X^{h_1}} g \,\mathrm{d}\mu^{h_1} = \frac{\displaystyle\int_X h_2 \mathrm{d}\mu}{\displaystyle\int_X h_1 \mathrm{d}\mu}.$$

[2] T^{h_2} and $(T^{h_1})^g$ are not defined on the same spaces but (X^{h_2},μ^{h_2}) and $((X^{h_1})^g,(\mu^{h_1})^g)$ are isomorphic in a very natural fashion, meaning that the subset $\{(h_1,h_2)=(i,j)\} \times \{k\}$ of X^{h_2} can be naturally assimilate to an explicit subset of $(X^{h_1})^g$, which is $(\{(h_1,h_2)=(i,j)\}\times\{k\})\times\{0\}$ if $k \leqslant i-1$, $(\{(h_1,h_2)=(i,j)\}\times\{i-1\})\times\{k-i+1\}$ if $i \leqslant k \leqslant j-1$ (see Figure 1.3), then we abusively consider that the spaces are the same.

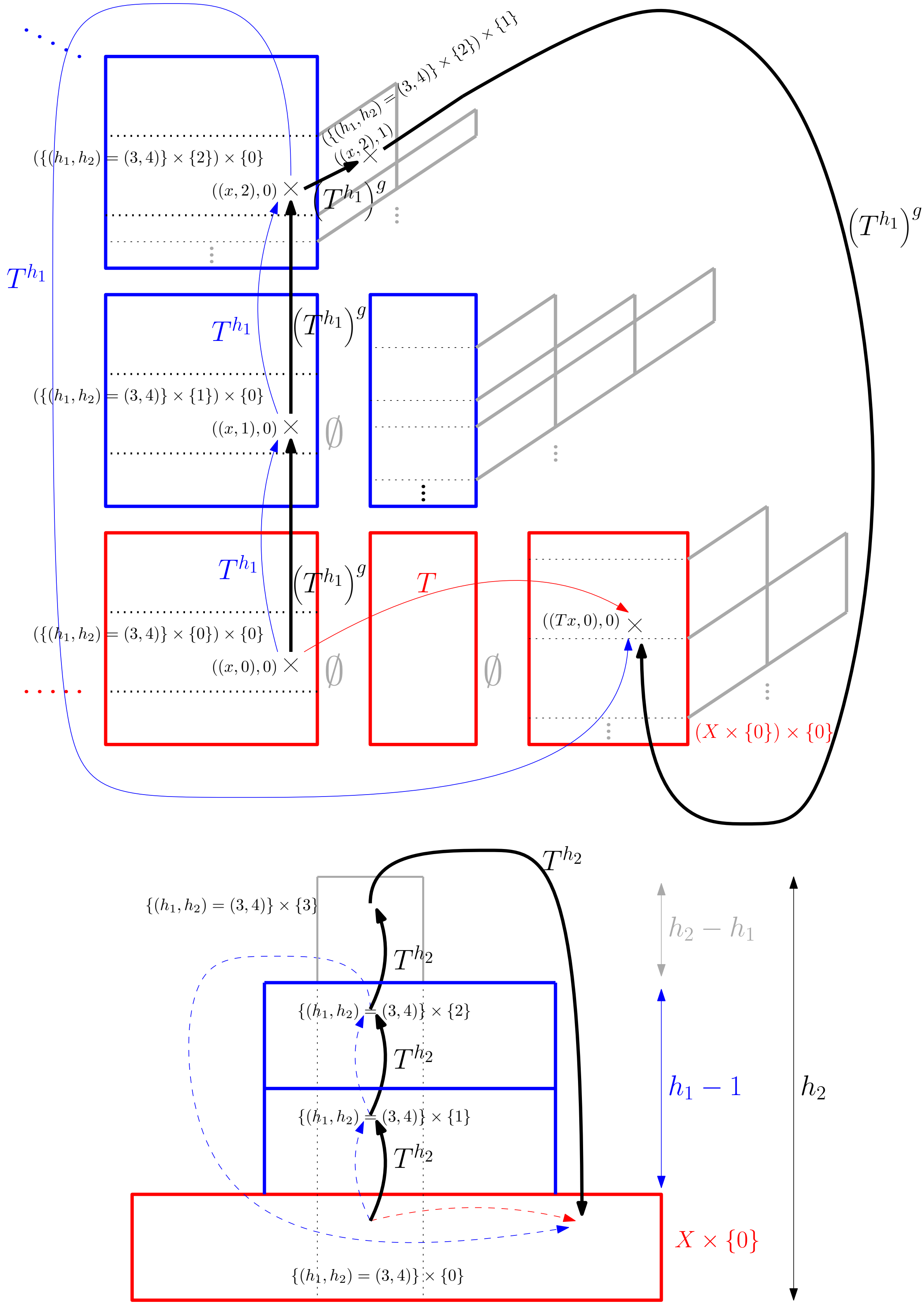


Figure 1: Drawing for Proposition 1.7-2

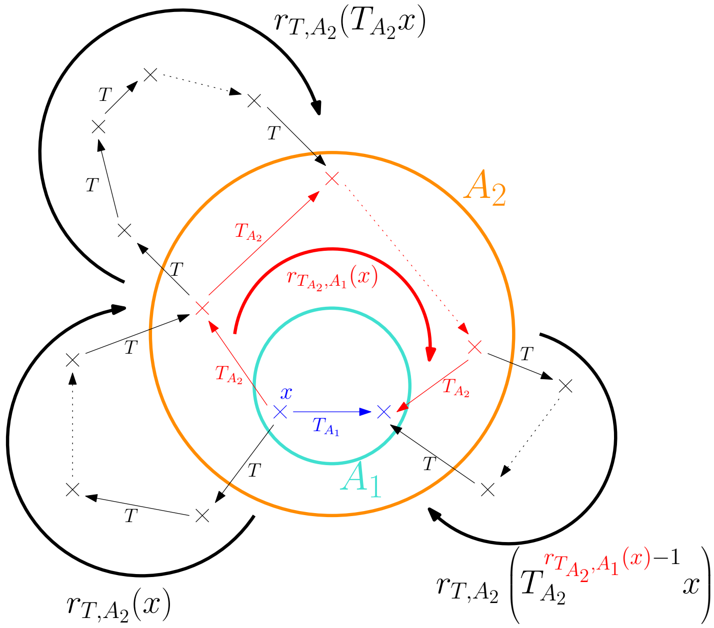


Figure 2: Drawing for Proposition 1.7-1

Proof of Proposition 1.7. We only give a proof for the integral of g, see Figures 1.3 and 1.3 for the other statements. We can compute the value of the integral using the definition of g. We can also find it without computation, using the equality $T^{h_2} = (T^{h_1})^g$. Indeed $\mu^{h_2}(X^1) = 1/\int_X h_2 \mathrm{d}\mu$, but it is also equal to $(\mu^{h_1})^g((X^1)^1) = \mu^{h_1}(X^1)/\int_{X^{h_1}} g \mathrm{d}\mu^{h_1}$ and $\mu^{h_1}(X^1)$ is equal to $1/\int_X h_1 \mathrm{d}\mu$. □

Proposition 1.7 states that if $A_1 \subset A_2$, then T_{A_1} is the induced map of T_{A_2} on an explicit subset of A_2 (which is A_1). The next proposition states that if we only know $\mu(A_1) < \mu(A_2)$, then T_{A_1} is isomorphic to the induced map of T_{A_2} on a subset of A_2 with the same measure as A_1.

There are the same ideas for towers. If we only know that $\int h_1 \mathrm{d}\mu < \int h_2 \mathrm{d}\mu$ (not necessarily $h_1 \leqslant h_2$), then T^{h_2} is isomorphic (not necessarily equal) to a tower of T^{h_1} and we know the integral of the height function, so that we know how much (X^{h_1}, T^{h_1}) is enlarged to obtain (X^{h_2}, T^{h_2}) up to conjugacy.

From now the transformations are assumed to be ergodic.

Proposition 1.8. *Let $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$ be an ergodic transformation. The following assertions hold true.*

1. *If A_1 and A_2 are subsets of X and $0 < \mu(A_1) < \mu(A_2)$, then T_{A_1} is isomorphic to $T_{A_1'}$ for some subset $A_1' \subset A_2$ with the same measure as A_1.*
2. *If $h_1, h_2 \colon X \to \mathbb{N}^*$ are two integrable functions and $\int_X h_1 \mathrm{d}\mu < \int_X h_2 \mathrm{d}\mu$, then T^{h_2} is isomorphic to $(T^{h_1})^g$ for some integrable function $g \colon X^{h_1} \mapsto \mathbb{N}^*$ satisfying*

$$\int_{X^{h_1}} g \mathrm{d}\mu^{h_1} = \frac{\displaystyle\int_X h_2 \mathrm{d}\mu}{\displaystyle\int_X h_1 \mathrm{d}\mu}.$$

Proof of Proposition 1.8. We first prove the equivalence between the assertions and then that the second one is true.

Assume the first point. Let h_1, h_2 as in the second point. T^{h_i} is isomorphic to $(T^{h_1+h_2})_{X^{h_i}}$. We can show that $\mu^{h_1+h_2}(X^{h_i}) = \int h_i \mathrm{d}\mu / \int (h_1+h_2)\mathrm{d}\mu$. Indeed we have $T^{h_1+h_2} = (T^{h_1})^g$ for some g by Proposition 1.7 and $X^{h_i} \subset X^{h_1+h_2}$ has the same measure as $(X^{h_1})^1 \subset (X^{h_1})^g$ and this is the inverse of $\int g \mathrm{d}\mu^{h_1}$ which is known by Proposition 1.7. Then we have

$$\mu^{h_1+h_2}(X^{h_1}) \leqslant \mu^{h_1+h_2}(X^{h_2}),$$

this implies that there exists $A \subset X^{h_2}$ with the same $\mu^{h_1+h_2}$-measure as X^{h_1} and such that $(T^{h_1+h_2})_{X^{h_1}}$ is isomorphic to $(T^{h_1+h_2})_A$ which is equal to $((T^{h_1+h_2})_{X^{h_2}})_A$. Then T^{h_1} is isomorphic to $(T^{h_2})_A$. Finally T^{h_2} is a tower of height $h := r_{T^{h_2},A}$ for $(T^{h_2})_A$ so by Proposition 1.5 it is isomorphic to $(T^{h_1})^g$ with g of the form $h \circ \varphi^{-1}$. Its μ^{h_1}-integral is the integral of the return time in A for T^{h_2}, then it is $1/\mu^{h_2}(A)$ and the result follows from the equality $\mu^{h_1+h_2}(A) = \mu^{h_1+h_2}(X^{h_2})\mu^{h_2}(A)$.

Conversely assume the second point. Let A_1, A_2 as in the first point. It is easy to check that $T^i T_{A_2} = T_{T^i(A_2)} T^i$ for every integer i. Then $T^i \colon A_2 \to T^i(A_2)$ is a conjugacy between T_{A_2} and $T_{T^i(A_2)}$. By ergodicity, $A := A_1 \cap T^i(A_2)$ is of positive measure for some i that we fix. T_{A_1} is isomorphic to $(T_A)^{h_1}$ with $h_1 = r_{T_{A_1},A}$. Similarly T_{A_2} is isomorphic to $(T_A)^{h_2}$ with $h_2 = r_{T_{A_2'},A}$. The μ_A-integral of h_i is $1/\mu_{A_i}(A) = \mu(A_i)/\mu(A)$, then

$$\int_A h_1 \mathrm{d}\mu_A < \int_A h_2 \mathrm{d}\mu_A$$

and there exists $g \colon A^{h_1} \to \mathbb{N}^*$ such that $(T_A)^{h_2}$ is isomorphic to $((T_A)^{h_1})^g$. Then T_{A_2} is isomorphic to $(T_{A_1})^{g'}$ with g' of the form $g \circ \varphi^{-1}$ by Proposition 1.5. We get T_{A_1} when we induce $(T_{A_1})^{g'}$ on $(A_1)^1$. By Proposition 1.5, the desired A_1' is of the form $\varphi'((A_1)^1)$, its μ_{A_2}-measure is the inverse of the μ^{h_1}-integral of g, i.e. $\mu(A_1)/\mu(A_2)$. Then $\mu(A_1') = \mu(A_1)$.

Now we prove the second point. By the ergodic theorem, there exists N such that for all $n \geqslant N$, we have

$$\sum_{0 \leqslant j < n} h_1(T^j x) < \sum_{0 \leqslant j < n} h_2(T^j x)$$

for every x in a subset B of positive measure. We can find a non null subset A of B such that $r_{T,A}$ is bounded below by N. By Proposition 1.7-1, T^{h_i} is isomorphic to $(T_A)^{h_i'}$ with

$$h_i'(x) = \sum_{0 \leqslant j < r_{T,A}(x)} h_i(T^j x).$$

Then we have $h_1' \leqslant h_2'$. By Proposition 1.7, there exists $g' \colon A^{h_1'} \to \mathbb{N}^*$ such that $(T_A)^{h_2'}$ is isomorphic to $((T_A)^{h_1'})^{g'}$. By Proposition 1.5, T^{h_2} is isomorphic to $(T^{h_1})^g$ for g of the form $g' \circ \varphi^{-1}$. Again by Proposition 1.7 applied to g', the $\mu^{X^{h_1}}$-integral of g is equal to $\int_A h_2' \mathrm{d}\mu_A / \int_A h_1' \mathrm{d}\mu_A$. The result follows since $\int_A h_i' \mathrm{d}\mu_A = \int_X h_i \mathrm{d}\mu / \mu(A)$. □

2 Kakutani equivalence

Definition 2.1. *Let $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$, $S \in \mathrm{Aut}(Y, \mathcal{B}, \nu)$ be two ergodic transformations.*

1. *T and S are said to be* ***Kakutani equivalent****, and we write $T \sim_K S$, if T_A and S_B are isomorphic for some $A \in \mathcal{A}$, $B \in \mathcal{B}$.*

2. *Moreover they are* ***even Kakutani equivalent****, and we write $T \sim_{eK} S$, if in addition two such measurable sets have the same measure : $\mu(A) = \nu(B)$.*

Proposition 2.2. *Kakutani equivalence and even Kakutani equivalence are equivalence relations.*

Proof of Proposition 2.2. $\sim_{\mathrm{K}}$ is obviously reflexive (take $A = B$) and symmetric. For transitivity, let $S \in \mathrm{Aut}(X, \mathcal{A}, \mu), T \in \mathrm{Aut}(Y, \mathcal{B}, \nu), U \in \mathrm{Aut}(Z, \mathcal{C}, \rho)$ be three ergodic transformations and $A \in \mathcal{A}$, $B_1, B_2 \in \mathcal{B}$, $C \in \mathcal{C}$ such that S_A is isomorphic to T_{B_1} and T_{B_2} is isomorphic to U_C. We use the same trick as in the proof of Proposition 1.8 : T_{B_2} is isomorphic to $T_{B_2'}$ where B_2' is some $T^i(B_2)$ whose intersection with B_1, denoted by B, is not a null set. Now we induce T_{B_1} and $T_{T^i(B_2)}$ on B and by Proposition 1.5 we have some conjugations $\varphi_1 \colon B_1 \to A$ and $\varphi_2 \colon T^i(B_2) \to C$ such that $S_{\varphi_1(B)}$ and $U_{\varphi_2(B)}$ are isomorphic.

Reflexivity and symmetry are also obvious for $\sim_{\mathrm{eK}}$. For transitivity, it will be the same proof as below but with the additional assumptions that $\mu(A) = \nu(B_1)$ and $\nu(B_2) = \rho(C)$. Since $\varphi_1 \colon B_1 \to A$ and $\varphi_2 \colon T^i(B_2) \to C$ are two measure isomorphisms, we have $\mu_A(\varphi_1(B)) = \nu_{B_1}(B)$ and $\nu_{T^i(B_2)}(B) = \rho_C(\varphi_2(B))$. Then the so-called additional assumptions imply $\mu(\varphi_1(B)) = \nu(B) = \rho(\varphi_2(B))$. ☐

Abramov's formula gives the entropy of an induced map.

Theorem 2.3 (Abramov's formula)**.** *Let $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$ be an ergodic transformation and A a measurable subset of X. Then*

$$\mu(A) h(T_A) = h(T).$$

Corollary 2.4. *Entropy is an invariant of even Kakutani equivalence.*

Remark 2.5. T is recovered by inducing T^h on X^1, so Abramov's formula can also be stated as follows :

$$h(T) = h(T^h) \int_X h \mathrm{d}\mu.$$

We end this section with another caracterisation of Kakutani equivalent, this is the definition given in [ORW82]. First we introduce some terminologies.

Definition 2.6. *Let $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$, $S \in \mathrm{Aut}(Y, \mathcal{B}, \nu)$. We say that T is derivative of S and S is a primitive of T, if T is isomorphic to an induced transformation of S, or equivalently if S is isomorphic to a tower of T.*

Example 2.7. T_A is a derivative of T, T is a derivative of T^h.

Proposition 2.8. *Let $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$, $S \in \mathrm{Aut}(Y, \mathcal{B}, \nu)$. The following assertions are equivalent.*

1. $T \sim_K S$*;*

2. T *and* S *have a common derivative;*

3. T *and* S *have a common primitive.*

Proof of Corollary 2.8. Assume $T \sim_{\text{K}} S$ and let A and B be subsets such that T_A and S_B are isomorphic. Then T is isomorphic to $(T_A)^{h_1}$ with $h_1 = r_{T,A}$ and S is isomorphic to $(S_B)^{h_2}$ with $h_2 = r_{S,B}$. By Proposition 1.5 $(S_B)^{h_2}$ is isomorphic to $(T_A)^{h_2'}$ for some h_2' of the form $h_2 \circ \varphi^{-1}$. Then T_A is a derivative of T and S.

Assume that U is a common derivative of T and S. Then T and S are respectively isomorphic to U^{h_1} and U^{h_2} for some height functions h_1 and h_2. By Proposition 1.7, $U^{h_1+h_2}$ is a tower of U^{h_1} and U^{h_2}, then it is a common primitive of T and S.

Assume that U is a common primitive of T and S. Then T and S are respectively isomorphic to U_A and U_B for some subsets A and B. Using the same trick as in the proof of Proposition 1.8, U_B is isomorphic to some $U_{T^i(B)}$ where $C := A \cap T^i(B)$ is of positive measure. Finally we induce both U_A and $U_{T^i(B)}$ on C, according to Propositon 1.5 it corresponds to inducing T and S on some subsets $\varphi_1(A)$ and $\varphi_2(A)$ and the induced transformations are isomorphic. □

3 Generalization to $\mathbb{Z}^d$-actions : M-Kakutani equivalence

In the sequel, we introduce a generalization of Kakutani equivalence for $\mathbb{Z}^d$-actions. Group actions will always be free, ergodic, bimeasurable and p.m.p.

3.1 Stable orbit equivalence

First we need to define the notion of stable orbit equivalence which is an orbit equivalence but not necessarily defined on the whole space or even onto.

Definition 3.1. *Let* G *and* H *be groups,* $T\colon G \curvearrowright (X, \mathcal{A}, \mu)$ *and* $S\colon H \curvearrowright (Y, \mathcal{B}, \nu)$ *be free ergodic p.m.p. actions.*

A ***stable orbit equivalence (SOE)*** *between* $(X, \mathcal{A}, \mu, T)$ *and* $(Y, \mathcal{B}, \nu, S)$ *is a measure isomorphism* $\varphi\colon (U, \mathcal{A}_U, \mu_U) \to (V, \mathcal{B}_V, \nu_V)$ *(with subsets* $U \subset X$ *and* $V \subset Y$ *of positive measure) satisfying*

$$\text{for } \mu\text{-a.e. } x \in X,\ \varphi(\text{Orb}_T(x) \cap U) = \text{Orb}_S(\varphi(x)) \cap V.$$

U *and* V *are respectively denoted by* $\text{dom}(\varphi)$ *and* $\text{rng}(\varphi)$*. The compression of* φ *is the constant*

$$\text{comp}(\varphi) := \frac{\nu(\text{rng}(\varphi))}{\mu(\text{dom}(\varphi))}.$$

If U *and* V *are of full measure, then* φ *is said to be an* ***orbit equivalence (OE)****.*

Notice that for $A \subset \mathrm{dom}(\varphi)$ and $\varphi_{|A} \colon A \to \varphi(A)$, we have $\mathrm{comp}(\varphi_{|A}) = \mathrm{comp}(\varphi)$. It is a consequence of the equality $\varphi_\star \nu_{\mathrm{rng}(\varphi)} = \mu_{\mathrm{dom}(\varphi)}$.

Remark 3.2. If the actions are free, then we can define partial cocycles (just cocycles if the SOE is an OE)

$$\begin{cases} \alpha \colon \{(g,x) \in G \times X : x \in U \cap T^{g^{-1}}(U)\} \to H \text{ and} \\ \beta \colon \{(h,y) \in H \times Y : y \in V \cap T^{h^{-1}}(V)\} \to G \end{cases}$$

given by the equations

$$\begin{cases} \forall x \in U \cap T^{g^{-1}}(U),\ \varphi(T^g x) = S^{\alpha(g,x)}(\varphi(x)) \text{ and} \\ \forall y \in V \cap T^{h^{-1}}(V),\ \varphi^{-1}(S^h y) = T^{\beta(h,y)}(\varphi^{-1}(y)). \end{cases}$$

The partial cocycles satisfy the cocycle identity

$$\forall g,k \in G,\ \forall x \in U \cap T^{k^{-1}}(U) \cap T^{(gk)^{-1}}(U),\ \alpha(gk,x) = \alpha(g,T^k x)\alpha(k,x)$$

and similarly for β.

Remark 3.3. For $G = H = \mathbb{Z}$, if φ is a SOE between ergodic transformations $T \in \mathrm{Aut}(X,\mathcal{A},\mu)$ and $S \in \mathrm{Aut}(Y,\mathcal{B},\nu)$, then it is an OE between T_U and S_V with $U = \mathrm{dom}(\varphi)$ and $V = \mathrm{rng}(\varphi)$, and conversely. Indeed we have $\mathrm{Orb}_T(x) \cap U = \mathrm{Orb}_{T_U}(x)$ for every $x \in U$, and similarly for S and B.

If $\varphi \colon U \to V$ is a conjugacy between T_U and S_V, then it is an SOE between T and S with cocycles $\alpha(1,x) = r_{S,V}(\varphi(x))$ and $\beta(1,y) = r_{T,U}(\varphi^{-1}(y))$. Indeed for every $x \in U \cap T^{-1}U$, we have

$$\varphi(Tx) = \varphi(T_U x) = S_V(\varphi(x)) = S^{r_{S,V}(\varphi(x))}\varphi(x)$$

and similarly for the other cocycle.

3.2 M-Kakutani equivalence

Now we define a binary relation $\overset{M}{\rightsquigarrow}$ called "M-Kakutani equivalence", for a $d \times d$ matrix M, among free ergodic $\mathbb{Z}^d$-actions. We will prove that the equivalence relation generated by all the M-Kakutani equivalences is exactly Kakutani equivalence (up to flip-conjugacy) in the case $d = 1$. Then these notions allow us to define Kakutani equivalence in higher dimension.

In $\mathbb{R}^d$, consider the norm $\|.\|$ defined by $\|v\| = \max\{|v_i| \mid 1 \leqslant i \leqslant n\}$.

Definition 3.4. *Let $d \geqslant 1$ and M a $d \times d$ real matrix. M-Kakutani equivalence is a binary relation $\overset{M}{\rightsquigarrow}$ defined as follows. Given free ergodic $\mathbb{Z}^d$-actions T on $(X,\mathcal{A},\mu)$ and S on $(Y,\mathcal{B},\nu)$, we write $T \overset{M}{\rightsquigarrow} S$ if there exists a SOE φ between T and S, with*

$\mathrm{dom}(\varphi) = X$, *such that for any* $\varepsilon > 0$ *there are* $N_\varepsilon > 0$ *and* $X_\varepsilon \subset X$ *of measure greater than* $1 - \varepsilon$ *satisfying for all* $x, y \in X_\varepsilon$ *belonging to the same* T*-orbit,*

$$\|\overrightarrow{T}(x,y)\| \geqslant N_\varepsilon \Rightarrow \|M\overrightarrow{T}(x,y) - \overrightarrow{S}(\varphi(x), \varphi(y))\| \leqslant \varepsilon \|\overrightarrow{T}(x,y)\|$$

where $\overrightarrow{T}(x,y)$ *(the* T*-vector from* x *to* y*) denotes the element* $k \in \mathbb{Z}^d$ *such that* $T^k x = y$ *and similarly for* $\overrightarrow{S}$*.*

For $d = 1$ *and* $M = (m)$*, we write* $T \overset{m}{\rightsquigarrow} S$*.*

Replacing the variable y by $T^k x$ for a variable $k \in \mathbb{Z}^d$, the end of the definition can be written as follows :

$$\text{"for all } x \in X_\varepsilon,\ k \in \mathbb{Z}^d \text{ with } T^k x \in X_\varepsilon,$$

$$\|k\| \geqslant N_\varepsilon \Rightarrow \|Mk - \overrightarrow{S}(\varphi(x), \varphi(T^k x))\| \leqslant \varepsilon \|k\|\text{"}.$$

Intuitively this means that one of the partial cocycles (see Remark 3.2) is almost linear.

It is not difficult to prove that if $T \overset{M_1}{\rightsquigarrow} S$ and $S \overset{M_2}{\rightsquigarrow} U$, then $T \overset{M_2 M_1}{\rightsquigarrow} U$. Therefore $\overset{I_d}{\sim}$ is transitive.

Proposition 3.5 ([JR84])**.** *If* φ *is a SOE given by the definition of* $T \overset{M}{\rightsquigarrow} S$*, then*

$$\mathrm{comp}(\varphi) = \nu(\mathrm{rng}(\varphi)) = \frac{1}{|\det M|}.$$

With the setting of the last proposition, this implies :

1. $|\det M| \geqslant 1$ (in particular M is invertible);
2. φ is an OE if and only if $|\det M| = 1$;
3. if $|\det M| = 1$, then $T \overset{M}{\rightsquigarrow} S$ if and only if $S \overset{M^{-1}}{\rightsquigarrow} T$;
4. I_d-Kakutani equivalence is an equivalence relation.

Proof of Proposition 3.5. Let $\varepsilon > 0$ and N_ε, X_ε as in the definition. Denote by B_n the set of vectors of norm $\|.\|$ less or equal to n, and $F_n := (B_n) \cap \mathbb{Z}^d$. Given $x \in X_\varepsilon$, for every $u \in \mathbb{Z}^d$ satisfying $T^u x \in X_\varepsilon$, define $v(u) := \overrightarrow{S}(\varphi(x), \varphi(T^u x))$. T is a free action so v is injective. When $\|u\| \geqslant N_\varepsilon$, $v(u)$ satisfies by definition $\|Mu - v(u)\| \leqslant \varepsilon \|u\|$. If in addition u is in B_n, then $v(u)$ is in $MB_n + \varepsilon B_n$. This last set is included in $\alpha(\varepsilon) M B_n$ for some quantity $\alpha(\varepsilon) > 1$ tending to 1 as ε tends to 0. Thus, by injectivity of $v(.)$, we have

$$|\{u \in F_n \backslash B_{N_\varepsilon} \mid T^u x \in X_\varepsilon\}| \leqslant |\{v \in G_n \mid S^v(\varphi(x)) \in \varphi(X_\varepsilon)\}|$$

with $G_n := (\alpha(\varepsilon) M B_n) \cap \mathbb{Z}^d$. $(F_n)_n$ and $(G_n)_n$ are Følner sequences of the group $\mathbb{Z}^d$. Then, taking x such that the ergodic theorem holds, the right hand side is equivalent to $\mu(X_\varepsilon)|F_n|$ and the left hand side to $\nu(\varphi(X_\varepsilon))|G_n|$. It is known that $|G_n|$ is equivalent to $|\det \alpha(\varepsilon) M| \times |F_n|$. Finally, ε is arbitrary and this gives

$$\mu(X) \leqslant |\det M| \nu(\varphi(X))$$

and the invertibility of M. The reverse inequality is shown similarly (for every v such that $S^v(\varphi(x)) \in \varphi(X_\varepsilon)$, define $u(v) := \vec{T}(x, \varphi^{-1} S^v \varphi(x)))$, $u(.)$ is injective and if v is in $B_n \backslash B_{N_\varepsilon}$, then $u(v)$ is in some set slightly larger than $M^{-1} B_n$, etc). ☐

Now we explain the link between M-Kakutani equivalence and Kakutani equivalence for $\mathbb{Z}$-actions.

Theorem 3.6. *Let $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$ and $S \in \mathrm{Aut}(Y, \mathcal{B}, \nu)$ be ergodic transformations, m a real number satisfying $|m| \geqslant 1$. The sign of m is denoted by* $\mathrm{sgn}(m)$*.*

Then $T \overset{m}{\rightsquigarrow} S$ if and only if $(T^{\mathrm{sgn}(m)})_A$ is isomorphic to S_B for some subsets A and B with $\nu(B)/\mu(A) = 1/|m|$.

This implies :

1. the equivalence relation generated by the m-Kakutani equivalences for $m \geqslant 1$ is exactly Kakutani equivalence (this is flip-Kakutani equivalence when the negative m are also considered);
2. 1-Kakutani equivalence and even Kakutani equivalence are the same relations;

For an aperiodic transformation $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$, we define a total order $<_T$ on each T-orbits :

$$x <_T y \iff \exists n > 0,\ T^n x = y.$$

We also define the intervals :

$$[x, y]_T := \begin{cases} \{z \mid x \leqslant_T z \leqslant_T y\} & \text{if } x \leqslant_T y \\ \{z \mid y \leqslant_T z \leqslant_T x\} & \text{if } y \leqslant_T x \end{cases}.$$

Proof of Theorem 3.6. Since $\overrightarrow{T^{-1}}(x, y) = -\vec{T}(x, y)$, it suffices to prove the result for $m \geqslant 1$.

Assume $T \overset{m}{\rightsquigarrow} S$. Let φ be the SOE given by the definition. Let $0 < \varepsilon < 1$ and N_ε, X_ε as in the definition. A subset A of X_ε can be found such that $\mu(A) > 0$ and $A, T(A), \ldots, T^{N_\varepsilon - 1}(A)$ are pairwise disjoint. Now the goal is to prove $T_A = \varphi^{-1} S_B \varphi$ with $B = \varphi(A)$, the equality $\mu(A) = m\nu(B)$ will follow since the SOE has compression $1/m$. By the property satisfied by A, we have $\vec{T}(x, T_A x) \geqslant N_\varepsilon$ for every $x \in A$. This implies $|m\vec{T}(x, T_A x) - \vec{S}(\varphi(x), \varphi(T_A x))| \leqslant \varepsilon |\vec{T}(x, T_A x)|$ and in particular $\vec{S}(\varphi(x), \varphi(T_A x))$ is positive. Then $(\varphi(T_A^i x))_{i \in \mathbb{Z}}$ is an $<_S$-increasing sequence. Moreover $\{\varphi(T_A^i x) \mid i \in \mathbb{Z}\}$ is exactly $Orb_S(\varphi(x)) \cap B$. Since $S_B(\varphi(x))$ is the $<_S$-least element in $Orb_S(\varphi(x)) \cap B$ which is $<_S$-greater than $\varphi(x)$, we have $S_B(\varphi(x)) = \varphi(T_A x)$.

Assume that there is a conjugation $\varphi \colon A \to B$ between T_A and S_B, with subsets A and B satisfying $\nu(B)/\mu(A) = 1/|m|$. In particular φ is an SOE (see Remark 3.3). Let V be a subset of Y satisfying $\nu(V) = 1/|m|$ and $B \subset V$. Then φ can be extended to a SOE of domain X and range V (see Proposition 2.7 in [Fur99]), the extension is also denoted by φ and the goal is to show that it is a suitable SOE to show $T \overset{m}{\rightsquigarrow} S$. Let

$\varepsilon > 0$. For every $x \in X$, denote by $z_T(x)$ the $<_T$-greatest element of $Orb_T(x) \cap A$ which is $<_T$-less than x. For every $n \geqslant 0$, $k_n(x)$ denotes the cardinality of $[x, T^n x]_T \cap A$, i.e.

$$k_n(x) = \sum_{i=0}^{n} \mathbb{1}_{T^i x \in A} \underset{n \to +\infty}{\sim} n\mu(A),$$

then it diverges to $+\infty$ as $n \to +\infty$. The ergodic theorem implies that for almost every $x \in X$, $\sum_{i=0}^{k_n(x)-1} r_{S,B}\left(S_B^i(\varphi(z_T(x)))\right) \sim k_n(x) \int_B r_{S,B} \mathrm{d}\nu_B = k_n(x)/\nu(B)$. Then for $\alpha > 0$ to be chosen later, there exists $N(x) > 0$ such that for every $n \geqslant N(x)$,

$$(1-\alpha)mn \leqslant \sum_{i=0}^{k_n(x)-1} r_{S,B}\left(S_B^i(\varphi(z_T(x)))\right) \leqslant (1+\alpha)mn.$$

Choose $N_\varepsilon > 0$ such that $X_\varepsilon := \{N(x) \leqslant N_\varepsilon\} \cap \left\{\left|\vec{S}(\varphi(x), \varphi(z_T(x)))\right| \leqslant \alpha m N_\varepsilon\right\}$ has measure greater than $1 - \varepsilon$. Let x and $T^n x$ in X_ε, with $n \geqslant N_\varepsilon$. The elements in $[z_T(x), T^n x]_T \cap A$ are exactly

$$z_T(x), T_A(z_T(x)), \ldots, T_A^{k_n(x)}(z_T(x)),$$

and $T_A^{k_n(x)}(z_T(x))$ is equal to $z_T(T^n x)$. Then we have

$$\begin{aligned} \vec{S}(\varphi(x), \varphi(T^n x)) = \quad & \vec{S}(\varphi(x), \varphi(z_T(x))) \\ + \quad & \sum_{i=0}^{k_n(x)-1} \vec{S}\left(\varphi(T_A^i(z_T(x))), \varphi(T_A^{i+1}(z_T(x))\right) . \\ + \quad & \vec{S}(\varphi(z_T(T^n x)), \varphi(T^n x)) \end{aligned}$$

Using $\varphi T_A = S_B \varphi$ and $\vec{S}\left(S_B^i y, S_B^{i+1} y\right) = r_{S,B}(S_B^i y)$ for every $y \in B$ (especially applied for $y = \varphi(z_T(x)))$, we finally obtain

$$|\vec{S}(\varphi(x), \varphi(T^n x)) - mn| \leqslant \alpha mn + 2\alpha m N_\varepsilon \leqslant 3\alpha mn.$$

We choose $\alpha = \varepsilon/(3m)$ and this concludes the proof for $n \geqslant N_\varepsilon$ since we have $n = \vec{T}(x, T^n x)$. For $n \leqslant -N_\varepsilon$, notice that $\vec{S}(\varphi(x), \varphi(T^n x)) = -\vec{S}(\varphi(y), \varphi(T^{|n|} y))$ with $y = T^n x$ and apply what has been done with y and $|n|$. □

According to del Junco and Rudolph [JR84], the following result is due to Nadler (unpublished work).

Theorem 3.7. *Let T and S be $\mathbb{Z}^d$-actions on $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ respectively. If $T \overset{M}{\rightsquigarrow} S$, then*

$$h(S) = \frac{h(T)}{|\det\ M|}.$$

For $d = 1$, this result is a consequence of Abramov's formula and Proposition 2.2.

The SOE hidden in the hypothesis $T \overset{M}{\rightsquigarrow} S$ of the last theorem has full domain. For some SOE not necessarily having full domain, Austin gave a similar result.

Theorem 3.8 ([Aus16]). *Let T and S be $\mathbb{Z}^d$-actions on $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ respectively. If φ is a SOE$_\infty$ or a SSOE$_1$ between these actions, with* $\mathrm{dom}(\varphi)\subset X$ *and* $\mathrm{rng}(\varphi)\subset Y$*, then*

$$\frac{h(T)}{\mu(\mathrm{dom}(\varphi))}=\frac{h(S)}{\nu(\mathrm{rng}(\varphi))}.$$

See [Aus16] for the definitions of SOE$_\infty$ (bounded stable orbit equivalence) and SSOE$_1$ (integrable semi-stable orbit equivalence). For a brief definition, a SOE$_\infty$ is a SOE with bounded partial cocycles and a SOE is a SSOE$_1$ if the partial cocycles can be extended to integrable full cocycles (by "cocycle" we mean that the extension satisfies the cocycle identity). Austin showed that SOE$_\infty$ implies SSOE$_1$ (the bounded partial cocycle can be extended to a bounded full cocycle, in particular this extension is integrable). Finally he showed that SSOE$_1$ implies $T \overset{M}{\rightsquigarrow} S$ or $S \overset{M}{\rightsquigarrow} T$ for some matrix M, depending on whether the compression is less or greater than 1, in the first case any extension to a SOE of full domain satisfies the definition of M-Kakutani equivalence (we can find similar ideas between this proof and the one of Theorem 3.6). Finally the SSOE$_1$ has the same compression as its extension, then it is equal to $1/|\det M|$ and Theorem 3.8 follows from Theorem 3.7.

4 A brief overview of Ornstein's theory and a parallel theory for Kakutani equivalence

Given a finite partition $\mathcal{P}=(P_1,\dots,P_d)$ on X and $T\in \mathrm{Aut}(X,\mathcal{A},\mu)$, one can associate for every point x of the space a word $(a_i)_{i\in\mathbb{Z}}$ where a_i is the integer in $\{1,\dots,d\}$ such that T^ix is in P_{a_i}. When studying a dynamic, this coding (for relevant partitions) brings a lot of information. We can also compare points by comparing the associate subwords.

Definition 4.1. *We define the normalized Hamming metric between words of same length by :*

$$d_n((a_i)_{1\leqslant i\leqslant n},(b_i)_{1\leqslant i\leqslant n})=\frac{1}{n}|\{1\leqslant i\leqslant n \mid a_i\neq b_i\}|.$$

This metric is used by Ornstein [Orn74] to compare finite subwords. Then he defined classes of transformations called "finitely determined" (FD) and "very weak Bernoulli" (VWB), these classes are equal (Ornstein in [Orn74] for one inclusion, Ornstein and Weiss in [ORW82] for the other) and the idea behind the definitions is that these transformations admit similar partitions giving close words (for the d-metric). Finally Ornstein showed that two such transformations with equal entropy are isomorphic. Bernoulli shifts are finitely determined, then entropy is a total invariant of conjugacy in this subclass.

The problem with the d-metric is that the words *abababa* and *bababab* for distinct letters a and b are not d-close whereas they both admits the same long subsequence *bababa*. Then we define another metric denoted by f, this is a more flexible version of d-metric.

Definition 4.2. *The f-metric between words of same length is defined by :*

$$f_n((a_i)_{1\leqslant i\leqslant n}, (b_i)_{1\leqslant i\leqslant n}) = 1 - \frac{k}{n}$$

where k is the maximal integer for which we can find equal subsequences $(a_{i_\ell})_{1\leqslant \ell\leqslant k}$ and $(b_{j_\ell})_{1\leqslant \ell\leqslant k}$, with $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$.

In [ORW82], Ornstein, Rudolph and Weiss replaced the $\bar{d}$-metric by the f-metric and then obtained an analogous theory for Kakutani equivalence. Indeed they defined a class of transformations called "finitely fixed" (FF) and showed that this coincides with the notion of "loosely Bernoulli" (LB) already defined by Feldman in [Fel76]. They finally stated the following equivalence theorem.

Theorem 4.3. *Two finitely fixed transformations with equal entropy are even Kakutani equivalent.*

The FF class contains the FD class (and then the Bernoulli shifts). Rank one transformations (odometers, irrational rotations, etc) are FF. It is not the unique result of the theory, other important results :

- the FF class (or equivalently LB) is closed under Kakutani equivalence, meaning that if T is FF and $T \sim_{\mathrm{K}} S$, then S is also FF[3];
- if T is FF, then so are any induced map and any tower.

In higher dimension, some authors generalized this theory for I_d-Kakutani equivalence (see for instance [Has92], [JŞ98], [JŞ01]).

Now we assume Theorem 4.3 and we will give important consequences that we can proove with the tools developped in last sections.

Corollary 4.4. *If both $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$ and $S \in \mathrm{Aut}(Y, \mathcal{B}, \nu)$ are FF and $h(S) > h(T) > 0$, then S is isomorphic to T_A with some subsets A with $\mu(A) = h(T)/h(S)$.*

> **Example 4.5.** Given a Bernoulli shift T, any other Bernoulli shift S of higher entropy can be recovered by inducing T.

Proof of Corollary 4.4. By Abramov's formula, We can find a subset $A' \subset X$ such that S and $T_{A'}$ have equal entropy (with $\mu(A') = h(T)/h(S)$). Then by Theorem 4.3 these transformations are even Kakutani equivalent, meaning that T_B and S_C are isomorphic for some subsets $B \subset A'$ and $C \subset Y$ satisfying

$$\mu_{A'}(B) = \nu(C).$$

In particular we have $\mu(B) < \nu(C)$, then $\int_X r_{T,B} \mathrm{d}\mu > \int_Y r_{S,C} \mathrm{d}\nu$. By Proposition 1.8, this implies that $(T_B)^{r_{T,B}}$ and $((S_C)^{r_{S,C}})^g$ are isomorphic for some g of $\mu^{r_{S,C}}$-integral equal to $\mu(C)/\mu(B) = 1/\mu(A')$. Then T is isomorphic to S^g, i.e. T_A and S are isomorphic for some subset A satisfying $\mu(A) = \mu(A')$. □

[3] In [Fer97], Ferenczi defines the class of LB transformations to be the smallest class which contains all the irrationnal rotations and closed under Kakutani equivalence.

Corollary 4.6. *If both $T \in \mathrm{Aut}(X, \mathcal{A}, \mu)$ and $S \in \mathrm{Aut}(Y, \mathcal{B}, \nu)$ are FF and $h(S) = h(T) = 0$, then for any $\varepsilon > 0$, S is isomorphic to T_A for some subset A with $\mu(A) = 1-\varepsilon$.*

> **Example 4.7.** Irrationnal rotations can be recovered by inducing any rank-one systems on arbitrary large subsets.

Proof of Corollary 4.6. For any subset $A' \subset X$, we have $h(S) = h(T_{A'})$. The last proof shows that consequently S is isomorphic to T_A for some subset A satisfying $\mu(A) = \mu(A')$. □

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